1. This problem is about the Bogoliubov transformation. A common tool in studying many-body quantum systems is the operator transform. Suppose the particle creation and annihilation operators \( a^\dagger_i \) and \( a_i \) can be algebraically expressed in terms of a new set of operators \( b^\dagger_i \) and \( b_i \) that obey the same canonical commutation relations:

\[
[b_i, b_j] = [b_i^\dagger, b_j^\dagger] = 0 \quad [b_i, b_j^\dagger] = \delta_{ij} .
\]  

The operators \( b^\dagger_i \) and \( b_i \) are often said to create/annihilate quasiparticles. The commutation relations, Eq. (1), imply that there is a unique state \( |B_i\rangle \) that is annihilated by all \( b_i \); this state is usually referred to as the quasiparticle vacuum, the states of the form \( b^\dagger_i |0\rangle \) are the one-quasiparticle states, etc. Whenever the quasiparticles can be labeled by the same quantum numbers (e.g. \( \tilde{k} \)) as the original bosonic particles of the theory, it is often convenient to make a unitary operator transform:

\[
b_i = U a_i U^\dagger, \quad b_i^\dagger = U a_i^\dagger U^\dagger,
\]  

where \( U \) is a unitary operator in the Fock space, usually of the form \( \exp(X) \) for some anti-hermitian polynomial \( X \) in \( a_i \) and \( a_i^\dagger \).

(a) Show that the unitarity of \( U \) automatically guarantees that \( b_n \) and \( b_n^\dagger \) satisfy Eq. (1), and that the quasiparticle state \( |B\rangle = U |0\rangle \) is the quasiparticle vacuum.

(b) Verify that for \( X = \sum_n (c_n a_n^\dagger - c_n^* a_n) \), \( \exp(X) a_n \exp(-X) = a_n - c_n \). This transform is a c-number shift.

(c) Now let \( X = \sum_n \frac{1}{2} \eta_n (e^{i\lambda n} (a_n^\dagger)^2 - e^{-i\lambda n} (a_n)^2) \) \( (\eta_n \) and \( \lambda_n \) are real). Show that for this \( U = \exp(X) \), Eqs. (2) define a diagonal canonical transform:

\[
b_i = a_i \cosh \eta_i - e^{i\lambda_i} a_i^\dagger \sinh \eta_i, \quad b_i^\dagger = \cosh \eta_i a_i^\dagger - e^{-i\lambda_i} \sinh \eta_i a_i .
\]  

(d) In order to see the utility of the Bogoliubov transformation, consider the simple case of one creation/annihilation operator pair with \( \lambda = \pi \). We then have

\[
b = a \cosh \eta + a^\dagger \sinh \eta .
\]  

Use this transformation to obtain the eigenvalues of the following Hamiltonian:

\[
H = \hbar \omega a^\dagger a + \frac{1}{2} V(aa^\dagger + a^\dagger a) .
\]  

Also give the upper limit on \( V \) for which this can be done.

(e) Write down the ground state of the Hamiltonian above in terms of the number states \( a^\dagger a |n\rangle = n |n\rangle \).
2. This problem is about Pauli’s method of solving the hydrogen atom. For all spherically-symmetric potentials, discrete spectra of bound state energies have \((2l+1)\)-fold degeneracy mandated by the \(SO(3)\) symmetry — all states \(|l, m, n_r\rangle\) with the same \(l\) and \(n_r\) but different \(m\) have the same energy \(E(l, n_r)\). For most potentials, there is no further degeneracy — different combinations of \(l\) and \(n_r\) give different energies. However, there are two “accidentally degenerate” exceptions to that rule: the spherically-symmetric harmonic oscillator potential \(\hat{V} = \frac{1}{2}M\omega^2\hat{r}^2\), and the Coulomb potential \(\hat{V} = -e^2Z/\hat{r}\). In both cases the extra degeneracy is due to non-obvious conservation laws leading to unexpected enlargement of the symmetry group from the rotations-only \(SO(3)\) to \(SU(3)\) in the harmonic case and to \(SO(3) \times SO(3)\) in the Coulomb case. (We saw this in problem 1 of HW 3 for the case of the two-dimensional harmonic oscillator where \(SO(2)\) is enlarged to \(SU(2) \sim SO(3)\).

The unexpected conservation law in the Coulomb case is the Laplace-Runge-Lenz theorem generalized from classical to quantum mechanics. Classically, we define the Runge-Lenz vector \(\mathbf{K}\) as

\[
\mathbf{K} \equiv \mathbf{p} \times \mathbf{L} - e^2ZM\mathbf{n}_r,
\]

where \(M\) is the particle’s mass, \(\mathbf{L} \equiv \mathbf{r} \times \mathbf{p}\) is its angular momentum and \(\mathbf{n}_r \equiv \mathbf{x}/r\) is a unit vector pointing towards the particle. The Laplace-Runge-Lenz theorem states that for the Coulomb (Newton) potential, \(\mathbf{K}\) is a conserved quantity, \(i.e.,\) does not change with time.

(a) Prove the classical Laplace-Runge-Lenz theorem.

The definition, Eq. (6) implies that \(\mathbf{x} \cdot \mathbf{K} = \mathbf{L}^2 - e^2ZMr\) and hence \(r = \mathbf{L}^2/(|\mathbf{K}| \cos \phi + e^2ZM)\) where \(\phi\) is the angle between \(\mathbf{K}\) and \(\mathbf{x}\). Therefore, constancy of the Runge-Lenz vector implies that the classical orbits are conical sections of eccentricity \(\epsilon = \mathbf{K}/e^2ZM\); for \(\epsilon < 1\) the orbit is a closed ellipse whose pericenter lies in the direction pointed to by \(\mathbf{K}\).

In quantum mechanics we define the Runge-Lenz vector operator

\[
\hat{\mathbf{K}} \equiv \frac{1}{2} (\hat{\mathbf{p}} \times \hat{\mathbf{L}} - \hat{\mathbf{L}} \times \hat{\mathbf{p}}) - e^2ZM\hat{\mathbf{x}}\hat{r}^{-1}.
\]

(b) Verify that each of the component operators \(\hat{K}_i\) is hermitian and is conserved, \(i.e.\) commutes with the Hamiltonian

\[
\hat{H} = \frac{1}{2M}\hat{\mathbf{p}}^2 - e^2Z\hat{r}^{-1}.
\]

To find out the Lie algebra generated by the conserved operators \(\hat{L}_i\) and \(\hat{K}_i\), we need their commutation relations. We know that \([\hat{L}_i, \hat{L}_j] = i\hbar\varepsilon_{ijk}\hat{L}_k\).

(c) Show that

\[
[ \hat{K}_i, \hat{L}_j ] = i\hbar\varepsilon_{ijk}\hat{K}_k \quad [ \hat{K}_i, \hat{K}_j ] = -2M\hat{H} \cdot i\hbar\varepsilon_{ijk}\hat{L}_k.
\]

Now consider the subspace of the Hilbert space spanned by the bound states of the Hamiltonian. On this subspace let us define two vector operators \(\hat{Q}_+\) and \(\hat{Q}_-\):

\[
\hat{Q}_\pm \equiv \frac{\hat{\mathbf{L}}}{\sqrt{2}} \pm \frac{\hat{\mathbf{K}}}{\sqrt{8M\hat{H}}}. \quad (10)
\]
(d) Show that the six operators $\hat{Q}_i^\pm$ are hermitian, conserved and obey the $SO(3) \times SO(3)$ commutation relations:

$$[\hat{Q}_i^+, \hat{Q}_j^-] = i\hbar\epsilon^{ijk}\hat{Q}_k^+,$$
$$[\hat{Q}_i^-, \hat{Q}_j^+] = i\hbar\epsilon^{ijk}\hat{Q}_k^-, \quad [\hat{Q}_i^+, \hat{Q}_j^-] = 0.$$  \hspace{1cm} (11)

This $SO(3) \times SO(3)$ Lie algebra can be used to describe all bound states as $|q_+, m_+, q_-, m_-\rangle$ — simultaneous eigenstates of the $\hat{Q}_i^\pm$ and $\hat{Q}_j^\pm$ operators. However, this description is somewhat redundant:

(e) Verify that $\hat{K} \cdot \hat{L} = \hat{L} \cdot \hat{K} = 0$ and use this fact to show that all bound states have $\hat{Q}_2^- = \hat{Q}_2^+$ and hence $q_+ = q_-$. Therefore we can label the bound states of the Coulomb potential as $|q, m_+, m_-\rangle$; their energies depend only on $q$ and thus are $(2q+1)^2$-fold degenerate. To compute these energies:

(f) First, show that

$$\hat{K}^2 = (e^2 Z M)^2 + 2M \hat{H}(\hat{L}^2 + \hbar^2)$$  \hspace{1cm} (12)

(in classical mechanics, $K^2 = (e^2 Z M)^2 + 2M E L^2$.)

(g) Second, use Eqs. (10) and (12) to derive

$$2\hat{Q}_2^+ + 2\hat{Q}_2^- + \hbar^2 = \frac{(e^2 Z M)^2}{-2M \hat{H}}.$$  \hspace{1cm} (13)

(h) And, finally, use Eqs. (13) to show that the energy of the $|q, m_+, m_-\rangle$ bound state is

$$E_N = -\frac{M(e^2 Z)^2}{2\hbar^2(2q + 1)^2} \equiv -\frac{M(e^2 Z)^2}{2\hbar^2 N^2}.$$  \hspace{1cm} (14)

where $N \equiv 2q + 1$ is a positive integer, usually called the principal quantum number of the bound state.

(i) Show that for each value of the principal quantum number $N$, the orbital quantum number $l$ takes all integer values between zero and $N - 1$.

(Hint: Use $\hat{L} = \hat{Q}_+ + \hat{Q}_-$.) Also, argue that this means that in terms of $l$ and the radial quantum number $n_r$, $N = l + n_r + 1$, which implies that the spectrum of $N$ consists of all positive integers.
3. This problem is about time-dependent perturbation theory and its relation with time-independent perturbation theory.

(a) When the potential $V$ is time-independent, work out $\langle s|\tilde{T}(t,0)|s\rangle$ to second order and identify $\Delta^{(1)}$, $\Delta^{(2)}$ and the “wave-function renormalization” $Z_i$ in the expansion of

$$\langle s|\tilde{T}(t,0)|s\rangle = Z_i e^{-i\Delta E t/\hbar} + \text{rapidly oscillating terms}$$

$$= Z_i - \frac{i}{\hbar} \left( \Delta_i^{(1)} + \Delta_i^{(2)} \right) t + \frac{1}{2!} \left( -\frac{i}{\hbar} \Delta_i^{(1)} t \right)^2 + \vartheta(V^3)$$

and show that they agree with the results from time-independent perturbation theory, Eqs. (5.1.42), (5.1.44) and (5.1.48b) in Sakurai. Note that this identification is done in the $t \to \infty$ limit where rapidly oscillating terms are dropped. Explain why this identification works.

(b) Now consider a harmonic perturbation $V = V_0 \cos \omega t$. Work out the second-order energy shift. Does your expression recover the result from time-independent perturbation theory in the limit $\omega \to 0$? Explain your answer.

4. This problem is about scattering in one dimension. The Lippmann-Schwinger formalism can be applied to a one-dimensional transmission-reflection problem with a finite range potential, $V(x) \neq 0$ for $0 < |x| < a$ only.

(a) Suppose that we have an incident wave coming from the left: $\langle x|\phi \rangle = e^{ikx}/\sqrt{2\pi}$. How must we handle the singular $1/(E - H_0)$ operator if we are to have a transmitted wave only for $x > a$ and a reflected wave and the original wave for $x < -a$? Is the $E \to E + i\epsilon$ prescription still correct? Obtain an expression for the appropriate Green’s function and write an integral equation for $\langle x|\psi^{(+)} \rangle$.

(b) Consider the special case of an attractive $\delta$-function potential

$$V = -\left( \frac{\gamma \hbar^2}{2m} \right) \delta(x), \quad (\gamma > 0).$$

Solve the integral equation to obtain the transmission and reflection amplitudes.

(c) The one-dimensional $\delta$-function potential with $\gamma > 0$ admits one and only one bound state for any value of $\gamma$. Show that the transmission and reflection amplitudes you computed have bound-state poles at the expected positions when $k$ is regarded as a complex variable.