1. This problem is about Generating Functions. The Fibonacci sequence is defined by

$$a_{n+1} = a_n + a_{n-1} \qquad a_0 = a_1 = 1 \ . \tag{1}$$

Define the generating function

$$A(z) = \sum_{n=0}^{\infty} a_n z^n \ . \tag{2}$$

   (a) Find $A(z)$ explicitly.

   (b) Find an expression for a_n.

   (c) Find the limit as $n \to \infty$ of $\frac{a_{n+1}}{a_n}$.

2. This problem is about a Free Particle on a Torus. A torus may be represented as a square with periodic boundary conditions (say one side of length a and one side of length b). A particle which passes the right hand boundary immediately finds itself at the same height on the left hand side; a particle can similarly move immediately between top and bottom. Consider a free particle with velocity (v_1, v_2). Under what circumstances is its motion ergodic? Does the particle actually reach all points in the system in this case?

3. This problem is about Microcanonical Calculations.

   (a) Find the volume V of a sphere of m dimensions and radius R. The way to do this is to consider the integral

$$\int_{-\infty}^{\infty} d^m x \ e^{-\sum_{i=1}^{m} x_i^2} \ . \tag{3}$$

   First do the integral as a product of m integrals. Then do it in polar coordinates.

   (b) Consider a collection of N particles in a one-dimensional potential well. Each particle has potential energy Ax^2, and does not interact with the other particles. Define the (one-dimensional) volume of the system to be

$$V = 2\sqrt{\left\langle \sum_{i}^{N} x_i^2 \right\rangle} \ . \tag{4}$$

   Using the result of the previous section, calculate V in the microcanonical ensemble for a given energy E. Use the result to calculate P.

4. This problem is about <u>Pressure in the Canonical Ensemble</u>. Consider a collection of N particles in a spherical box of radius R (in three dimensions). The Hamiltonian for these particles is

$$H = \sum_i^N \frac{\vec{p}_i^{\,2}}{2m} + W(r_i) \ , \tag{5}$$

where $W(x)$ is a potential that depends only on the distance of each particle to the outside of the box, and is

$$W(r) \ = \ \begin{cases} -W_0 \log \frac{R-r}{a} & \text{for } R > r > R-a \\ 0 & \text{for } r > R \text{ or } r < R-a \end{cases} . \tag{6}$$

The constant $a < R$ gives the width over which the wall is repulsive.

(a) Find a function of $r_1 \ldots r_N$ that gives the instantaneous pressure on the walls of the sphere.

(b) Find the average value of this function in the canonical ensemble.

(c) Evaluate the free energy of this system in the canonical ensemble, calculate the pressure directly from the free energy, and compare with part (b).

(d) Take the limit $a/R \to 0$ and simplify the expression for the pressure. Is it familiar? Should it be?