943 QUANTUM MECHANICS - Homework #7. Due: Friday, May 7, 2004.

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1. This problem is about the orbital angular momentum operator, $\hat{L}_i = \epsilon_{ijk}\hat{X}_j\hat{P}_k$.

   (a) Using canonical commutation relations between components of $\hat{X}_i$ and $\hat{P}_i$, show that
   $$\left[\ \hat{X}_i, \hat{L}_j\ \right] \ = \ i\hbar\epsilon_{ijk}\hat{X}_k \quad \text{and} \quad \left[\ \hat{P}_i, \hat{L}_j\ \right] \ = \ i\hbar\epsilon_{ijk}\hat{P}_k\ . \tag{1}$$

   (b) Show that
   $$\left[\ \hat{L}_i, \hat{L}_j\ \right] \ = \ i\hbar\epsilon_{ijk}\hat{L}_k \quad \text{and} \quad \epsilon_{ijk}\hat{L}_j\hat{L}_k \ = \ i\hbar\hat{L}_i\ . \tag{2}$$

   (c) Define $\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$ and show that
   $$\left[\ \hat{L}_z, \hat{L}_\pm\ \right] \ = \ \pm\hbar\hat{L}_\pm \quad \text{and} \quad \left[\ \hat{L}_+, \hat{L}_-\ \right] \ = \ 2\hbar\hat{L}_z\ . \tag{3}$$

   (d) Show that in the spherical coordinate basis
   $$\begin{aligned} \hat{L}_z\ \Psi(r,\theta,\phi) \ &= \ -i\hbar\frac{\partial}{\partial\phi}\ \Psi(r,\theta,\phi)\ , \\ \hat{L}_\pm\ \Psi(r,\theta,\phi) \ &= \ -i\hbar\ e^{\pm i\phi}\left(\pm i\frac{\partial}{\partial\theta} \ - \ \frac{1}{\tan\theta}\frac{\partial}{\partial\phi}\right)\ \Psi(r,\theta,\phi)\ . \end{aligned} \tag{4}$$

   (e) Compute $\hat{L}_i\hat{L}_i\ \Psi(r,\theta,\phi)$ in the same basis.
   (Hint: use $\hat{L}_i\hat{L}_i = \hat{L}_z^2 + \frac{1}{2}\hat{L}_+\hat{L}_- + \frac{1}{2}\hat{L}_-\hat{L}_+$.)

2. This problem is about Schwinger's oscillator model of angular momentum. The $SO(3)$ rotation group has both single-valued and double-valued representations, corresponding to integral and half-integral values of j, respectively. Both kinds of representations become single valued in terms of the Spin(3) group (the double cover of $SO(3)$); Spin(3) is isomorphic to $SU(2)$. The $SU(2)$ picture of the spin group is more convenient for deriving the explicit rotation matrices, $\mathcal{D}^{(j)}_{m,m'}(\phi,\vec{n})$ for all representations (j). In this problem, we will construct the $\mathcal{D}^{(j)}_{m,m'}$ matrix elements as explicit polynomials of the matrix elements $U_{\alpha\beta}$ of the $SU(2)$ matrix $U(\alpha,\vec{n}) \equiv \exp(-i\frac{\alpha}{2}\vec{n}\cdot\vec{\sigma})$.

   Our starting point is a system of two independent harmonic oscillators whose creation and annihilation operators $\hat{a}^\dagger_+$, $\hat{a}^\dagger_-$,$\hat{a}_+$,$\hat{a}_-$ obey the canonical commutation relations
   $$[\ \hat{a}_\alpha, \hat{a}_\beta\ ] \ = \ 0 \ = \left[\ \hat{a}^\dagger_\alpha, \hat{a}^\dagger_\beta\ \right]\ , \quad \left[\ \hat{a}_\alpha, \hat{a}^\dagger_\beta\ \right] \ = \ \delta_{\alpha\beta}\ , \qquad \alpha,\beta \ = \ +,- \tag{5}$$
   and a trio of model angular momentum operators
   $$\hat{J}_i \ = \ \frac{\hbar}{2}\sum_{\alpha,\beta}\sigma_{i,\alpha\beta}\ \hat{a}^\dagger_\alpha\hat{a}_\beta\ , \tag{6}$$
   where $\sigma_{i,\alpha\beta}$ are matrix elements of the Pauli matrices σ_i.

   (a) Compute the commutators $\left[\ \hat{J}_i, \hat{a}_\alpha\ \right]$ and $\left[\ \hat{J}_i, \hat{a}^\dagger_\alpha\ \right]$.

(b) Verify that $\left[\hat{J}_i, \hat{J}_j\right] = i\hbar\epsilon_{ijk}\hat{J}_k$; it is this relation that allows us to treat the $\hat{J}_i$ as model angular momenta.

(c) Prove that

$$\hat{J}_i\hat{J}_i = \hbar^2\frac{\hat{N}}{2}\left(\frac{\hat{N}}{2} + 1\right), \quad \text{where} \quad \hat{N} \equiv \hat{a}_+^\dagger\hat{a}_+ + \hat{a}_-^\dagger\hat{a}_- . \tag{7}$$

(Hint: first express $\hat{J}_z$ and $\hat{J}_\pm$ explicitly in terms of $\hat{a}_\pm$ and $\hat{a}_\pm^\dagger$; then compute $\hat{J}_i\hat{J}_i = \hat{J}_z^2 + \frac{1}{2}\{\hat{J}_+, \hat{J}_-\}$.)

(d) Show that for this model the states with definite values of j and m are precisely the states with definite numbers of oscillatorial quanta n_+ and n_-. Specifically,

$$|j, m\rangle = |n_+ = j+m, n_- = j-m\rangle = ((j+m)!\,(j-m)!)^{-1/2}\left(\hat{a}_+^\dagger\right)^{j+m}\left(\hat{a}_-^\dagger\right)^{j-m}|0\rangle \tag{8}$$

where $|0\rangle$ is the ground state of the two-oscillator system.

(e) Now suppose that for some unitary operator $\hat{V}$,

$$\hat{V}|0\rangle = |0\rangle \quad \text{and} \quad \hat{V}\hat{a}_\alpha^\dagger\hat{V}^\dagger = \sum_\beta \hat{a}_\beta^\dagger U_{\beta\alpha} \tag{9}$$

where $U_{\beta\alpha}$ is an $SU(2)$ matrix. Show that the relations, eq. (9), inevitably lead to

$$\hat{V}|j, m\rangle = \sum_{m'} |j, m'\rangle\, \mathcal{D}^{(j)}_{m',m} \tag{10}$$

and compute the matrix elements $\mathcal{D}^{(j)}_{m',m}$ as polynomials of the matrix elements of U.

Notice that for $j = 1/2$ the $\mathcal{D}^{(1/2)}$ matrix is U. Therefore, this exercise gives us the $\mathcal{D}^{(j)}$ matrices for states of all angular momenta j in terms of the two-by-two matrix for the states of $j = 1/2$.

(f) Prove the following lemma: For any operator $\hat{B}$ and a finite set of operators $\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_N$ satisfying the commutation relations $\left[\hat{A}_n, \hat{B}\right] = \sum_{n'} \hat{A}_{n'}C_{n',n}$ where $C_{n',n}$ is a finite N-by-N matrix of c-numbers,

$$\exp(t\hat{B})\hat{A}_n\exp(-t\hat{B}) = \sum_{n'} \hat{A}_{n'}\left[\exp(tC)\right]_{n',n} . \tag{11}$$

(Hint: differentiate the left hand side of eq. (11) with respect to t and then solve the differential equation.)

(g) Now consider the rotation operators $\hat{R}(\phi, \vec{n}) = \exp(\frac{\phi}{i\hbar}n_i\hat{J}_i)$ generated by the angular momentum operators, eq. (6). Show that these rotation operators do satisfy eq. (9), and the $SU(2)$ matrix $U_{\alpha\beta}$ happens to be $\left[\exp(\frac{\phi}{2i\hbar}n_i\hat{\sigma}_i)\right]_{\beta\alpha}$.

(h) Finally, explain why the $\mathcal{D}^{(j)}$ matrices that you've computed in this problem would work for any physical system with a well-defined angular momentum, and not just for the Schwinger model of this problem.