943 QUANTUM MECHANICS - Homework #6. Due: Friday, April 23, 2004.

---

1. Evaluate the propagator $K(x'', t; x', t_0)$ for the one-dimensional simple harmonic oscillator potential using the path integral formulation of Quantum Mechanics.
   (Hint: the solution is given in Sakurai.)

2. This problem is about rotations in three dimensions. For a rotation by angle α around an axis of a unit vector $\vec{n}$, the rotation matrix is

$$R_{ij}(\alpha, \vec{n}) \;=\; \delta_{ij}\cos\alpha \;-\; \epsilon_{ijk}n_k\sin\alpha \;+\; n_i n_j(1-\cos\alpha)\;. \qquad (1)$$

   A product of two rotations $R(\alpha', \vec{n}') \cdot R(\alpha'', \vec{n}'')$ is itself a rotation by some angle α around some axis $\vec{n}$. To determine α and $\vec{n}$, we demand

$$R_{ij}(\alpha, \vec{n}) \;=\; R_{ik}(\alpha', \vec{n}') \cdot R_{jk}(\alpha'', \vec{n}'')\;, \qquad (2)$$

   and then substitute eq. (1) and solve for α and $\vec{n}$.

   (a) Show that to second order in α' and α'',

$$\alpha\vec{n} \;\simeq\; \alpha'\vec{n}' \;+\; \alpha''\vec{n}'' \;+\; \frac{1}{2}\alpha'\alpha''\vec{n}'\times\vec{n}''\;. \qquad (3)$$

   (b) Prove $trR \equiv R_{ii} = 1 + 2\cos\alpha$ and $\epsilon_{ijk}R_{jk} = -2n_i\sin\alpha$ and use these formulas to derive exact expressions for $\cos\alpha$ and $\vec{n}\sin\alpha$ in terms of α', α'', $\vec{n}'$ and $\vec{n}''$.

3. This problem is about Pauli matrices and the relation between abstract rotations in spin space and rotations in three dimensions.

   (a) Use results derived in Homework #1 to show that for any two vectors $\vec{a}$ and $\vec{b}$,

$$(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = (\vec{a}\cdot\vec{b})\mathbf{1} \;+\; i(\vec{a}\times\vec{b})\cdot\vec{\sigma}. \qquad (4)$$

   (b) Show that for any angle α and for any unit vector $\vec{n}$,

$$U(\alpha, \vec{n}) \equiv \exp(-i\frac{\alpha}{2}\vec{n}\cdot\vec{\sigma}) = \cos(\frac{\alpha}{2})\mathbf{1} - i\sin(\frac{\alpha}{2})\;\vec{n}\cdot\vec{\sigma}\;. \qquad (5)$$

   (c) Solve for α and $\vec{n}$ such that

$$U(\alpha, \vec{n}) \;=\; U(\alpha', \vec{n}')\;U(\alpha'', \vec{n}'')\;. \qquad (6)$$

   (d) The commutation relations between generators of a Lie algebra completely determine the products of their exponentials. Consider a vector of 2×2 hermitian matrices, $\vec{S} \equiv \frac{\hbar}{2}\vec{\sigma}$; commutation relations between its compontents are exactly the same as between the components $\hat{J}_i$ of the angular momentum operator. Since the definition of the U matrices amounts to $U(\alpha, \vec{n}) = \exp(\alpha\vec{n}\cdot\vec{S}/i\hbar)$, which is precisely analogous to $\hat{R}(\alpha, \vec{n}) = \exp(\alpha\vec{n}\cdot\vec{\hat{J}}/i\hbar)$, we expect the product rule for the U matrices to be exactly the same as for the unitary rotation operators $\hat{R}$, which in turn sould be indentical to the product rule for the R_{ij} rotation matrices given in eq. (2).
   Verify that a solution of eq. (6) is automatically a solution of eq. (2).
   Is the converse also true?