943 QUANTUM MECHANICS - Homework #5. Due: Friday, April 9, 2004.

---

1. Once again consider a one-dimensional asymmetric potential well

$$V(x) \;=\; \begin{cases} V_1 & \text{for } x < 0 \\ V_0 & \text{for } 0 < x < a \\ V_2 & \text{for } x > a \end{cases} \qquad \text{where } V_1 < V_0 > V_2 \;. \tag{1}$$

   Compute the transmission and the reflection coefficients T and R for the asymmetric potential barrier — same formula for $V(x)$ as in last homework, except now $V_1 < V_0 > V_2$. Consider 2 cases: $E > V_0$ (classically, certain transmission) and $E < V_0$ (classically, certain reflection). Also, explain what happens when the barrier is actually a well, *i.e.*, $V_0 < V_{1,2} < E$.

2. A free particle of mass M moving with energy E is incident from the left upon a potential barrier of the form

$$V(x) \;=\; \begin{cases} 0 & \text{for } |x| > d/2 \\ \frac{1}{2}g\left((d/2)^2 - x^2\right) & \text{for } |x| \le d/2 \end{cases} \tag{2}$$

   (a) Assuming that $E \ll gd^2/8$, estimate the barrier penetration probability. (Hint: use WKB).

   (b) State any assumptions that you made in arriving at your answer to part (a).

3. Consider a single particle of mass M, interacting with a 1-dimensional potential of the form

$$V(x) \;=\; -g\;[\;\delta(x+a) \;+\; \delta(x-a)\;]\;. \tag{3}$$

   In a world with $\hbar = M = 1$, what is the minimum value of g for which an odd-parity bound state exists?

4. This problem is about the Green's function

$$G_{k+i\epsilon}(\vec{x},\vec{x}\,') \;=\; \int_{-\infty}^{\infty}\frac{d^3q}{(2\pi)^3}\frac{e^{i\vec{q}\cdot(\vec{x}-\vec{x}\,')}}{(k-q+i\epsilon)(k+q+i\epsilon)}\;. \tag{4}$$

(a) Show that the Green's function $G_k(\vec{x},\vec{x}\,')$ is a solution of the differential equation

$$(\vec{\nabla}^2+k^2)G_k(\vec{x},\vec{x}\,') \;=\; \delta^3(\vec{x}-\vec{x}\,')\;. \tag{5}$$

(b) Write the Schrödinger equation

$$\left(-\frac{\hbar^2}{2M}\vec{\nabla}^2 \;+\; U(\vec{x})\right)\Psi(\vec{x}) \;=\; E\Psi(\vec{x}) \tag{6}$$

in the form

$$(\vec{\nabla}^2 \;+\; k^2)\Psi(\vec{x}) \;=\; V(\vec{x})\Psi(\vec{x})\;, \tag{7}$$

with $k^2 = 2ME/\hbar^2$ and $V(\vec{x}) = 2MU(\vec{x})/\hbar^2$, and show that

$$\Psi(\vec{x}) \;=\; \int_{-\infty}^{\infty} d^3x' G_k(\vec{x},\vec{x}\,')V(\vec{x}\,')\Psi(\vec{x}\,') \tag{8}$$

is a solution of this equation.

(c) Let

$$\phi_{\vec{k}}(\vec{x}) = \frac{1}{(2\pi\hbar)^{3/2}}e^{i\vec{k}\cdot\vec{x}} \tag{9}$$

denote the solution of the homogeneous equation $(\vec{\nabla}^2 \;+\; k^2)\phi_{\vec{k}}(\vec{x}) \;=\; 0$. Show that $\Psi(\vec{x}\,')$ which fulfills the integral equation

$$\Psi(\vec{x}) \;=\; \phi_{\vec{k}}(\vec{x}) \;+\; \int_{-\infty}^{\infty} d^3x' G_k(\vec{x},\vec{x}\,')V(\vec{x}\,')\Psi(\vec{x}\,') \tag{10}$$

is a solution of the Schrödinger equation.

(d) Let $\Psi^{\pm}_{\vec{k}}(\vec{x})$ be a solution of the integral equation

$$\Psi^{\pm}_{\vec{k}}(\vec{x}) \;=\; \phi_{\vec{k}}(\vec{x}) \;+\; \int_{-\infty}^{\infty} d^3x' G_{k\pm i0}(\vec{x},\vec{x}\,')V(\vec{x}\,')\Psi(\vec{x}\,') \tag{11}$$

where $G_{k\pm i0}$ means $\lim_{\epsilon\to 0} G_{k\pm i\epsilon}$ where the limit is to be taken after the integration (in eq. (4)) has been performed. Show that

i. $\Psi^{+}_{\vec{k}}(\vec{x})$ represents a plane wave and an outgoing spherical wave.

ii. $\Psi^{-}_{\vec{k}}(\vec{x})$ represents a plane wave and an incoming spherical wave.