943 QUANTUM MECHANICS - Homework #4. Due: Friday, March 26, 2004.

No collaboration permitted on this homework set.

---

1. This excercise is about the WKB approximation to the harmonic oscillator.

   (a) The WKB approximation leads to a corrected Bohr-Sommerfeld rule for the bound states' energies:

   $$\oint \vec{p} \cdot \vec{x} \ = \ 2\pi\hbar(n + \frac{1}{2}) \ . \tag{1}$$

   Show that for the Harmonic oscillator this rule gives exactly correct energies for all eigenstates.

   (b) Write down the WKB approximation for the oscillator's eigenstates, $\Psi_n(x)$. Do not bother normalizing $\Psi_n^{WKB}(x)$, but please describe them in both classically-allowed and classically-forbidden regions of space.

   (c) Explain why for large n, $\Psi_n^{WKB}(x) \approx \Psi_n(x)$, for both allowed and forbidden values of x.

   (d) To verify that this is indeed the case, compute the large-n asymptotics of $\Psi_n(x)$ and compare to $\Psi_n^{WKB}(x)$.

   Hint: Start from the lemma proven in Homework #2, equation

   $$\Psi_n(x) \ \propto \ \exp(M\omega x^2/2\hbar) \int_{-\infty}^{+\infty} dk k^n \exp(ixk - \frac{k^2\hbar}{4M\omega}) \ , \tag{2}$$

   and then substitute $x = \sqrt{(2n+1)\hbar/M\omega}\, y$ (note that classical turning points correspond to $y = \pm 1$), make a similar redefinition of k and rewrite eq. (2) in the form

   $$\Psi_n(x) \ \propto \ \int dt f(t) \exp\left((n + \frac{1}{2}) g(y,t)\right) \ . \tag{3}$$

   The techniques for obtaining the large-n limit of integrals like eq. 3 are described in the notes available on the course homepage next to this assignment.

2. Consider a one-dimensional asymmetric potential well

$$V(x) = \begin{cases} V_1 & \text{for } x < 0 \\ V_0 & \text{for } 0 < x < a \\ V_2 & \text{for } x > a \end{cases} \qquad \text{where } V_1 > V_0 > V_2 \,. \tag{4}$$

If the well were symmetric ($V_1 = V_2$), there would have to be at least one bound state even for a very-narrow well. Show that this is not the case for $V_1 \neq V_2$ — a narrow asymmetric well may have no bound states at all — and compute the lowest value of a for which the first bound state appears in the well's spectrum.

Hint: Use $\tan(\alpha + \beta) = (\tan\alpha + \tan\beta)/(1 - \tan\alpha\tan\beta)$.