943 QUANTUM MECHANICS - Homework #3. Due: Friday, March 5, 2004.

1. In a one-dimensional space, a free particle has a Hamiltonian $\hat{H} = \hat{\vec{P}}^2/2M$.
   (a) Write down the evolution operator $\hat{U}(t;t_0)$ of the free particle and compute its matrix elements in the coordinate representation. ($\hat{U}(x,t;y,t_0) \equiv \langle x|\,\hat{U}(t;t_0)\,|y\rangle$ is often called *the evolution kernel of the Schrödinger equation.*)
   (b) Suppose the initial quantum state of the particle is a Gaussian wave packet $\Psi(x,t_0) = \sqrt{\alpha}\pi^{-1/4}\exp(ikx - \alpha^2x^2/2)$. Compute $\Psi(x,t)$ for all future times t and show that the probability density keeps its Gaussian form $|\Psi(x,t)|^2 = \frac{1}{b\sqrt{\pi}}\exp\left(-(x-a)^2/b^2\right)$.
   (c) Write down the Gaussian parameters a and b as explicit functions of time and explain the physical meaning of their time dependence.

2. Next consider the one-dimensional harmonic oscillator, whose Hamiltonian is given by

$$\hat{H} = \frac{1}{2M}\hat{P}^2 + \frac{M\omega^2}{2}\hat{X}^2 . \tag{1}$$

   (a) For any quantum state of the oscillator, the expectation values $\left\langle\hat{X}\right\rangle_\Psi$ and $\left\langle\hat{P}\right\rangle_\Psi$ obey classical equations of motion, *i.e.* they both oscillate with frequency ω. Show that this implies that the stationary-basis matrix elements $\langle m|\,\hat{X}\,|n\rangle$ and $\langle m|\,\hat{P}\,|n\rangle$ must vanish unless $E_m - E_n = \pm\hbar\omega$.
   (b) Compute $\langle m|\,\hat{X}\,|n\rangle$ and $\langle m|\,\hat{P}\,|n\rangle$ using the explicit coordinate-space wave functions $\Psi_n(x)$ given in Homework #2.

3. Consider a harmonic oscillator subject to a time-dependent external force; now the Hamiltonian is

$$\hat{H} \;=\; \frac{1}{2M}\hat{P}^2 \;+\; \frac{M\omega^2}{2}\hat{X}^2 \;-\; F(x)\hat{X} \;\;. \tag{2}$$

Among the solutions to the time-dependent Schrödinger equation, $i\hbar\partial\Psi(x,t)/\partial t = \hat{H}(t)\Psi(x,t)$ are the so-called *coherent states*

$$\Psi_{coh}(x,t) \;=\; \left(\frac{M\omega}{\pi\hbar}\right)^{1/4} \exp\frac{1}{\hbar}\left[-\frac{1}{2}M\omega(x-\bar{x}(t))^2 + i\bar{p}(t)(x-\bar{x}(t)) + is(t)\right] \tag{3}$$

that retain their Gaussian form at all times

(a) Compute $\langle\hat{X}\rangle$, $\langle\hat{P}\rangle$, $\Delta\hat{X}$ and $\Delta\hat{P}$ for a coherent state and compare their values to the $\langle\hat{X}\rangle$, $\langle\hat{P}\rangle$, $\Delta\hat{X}$ and $\Delta\hat{P}$ obtained for the ground state of the free oscillator.

(b) Show that the coherent states are indeed solutions to the time-dependent Schrödinger equation, provided $\bar{x}(t)$ and $\bar{p}(t)$ obey the classical equations of motion

$$\frac{d\bar{x}}{dt} \;=\; \frac{\bar{p}}{M}, \quad \frac{d\bar{p}}{dt} \;=\; F(t) \;-\; \omega^2 M\bar{x} \tag{4}$$

and the phase factor $s(t)/\hbar$ is given by the action integral of the classical forced oscillator

$$ds(t) \;=\; \bar{p}(t)d\bar{x}(t) \;-\; (H_{cl}(\bar{x},\bar{p},t) + \frac{1}{2}\hbar\omega)dt \tag{5}$$

where H_{cl} is the classical Hamiltonian (the $\hbar\omega/2$ term added to H_{cl} is the zero-point energy of the quantum oscillator).

4. Finally, consider a charged spinless particle in a constant magnetic field. For simplicity, assume that the particle is free to move in the xy plane but not perpendicular to that plane, and that the magnetic field is in the z-direction. The classical Hamiltonian of such a particle is given by

$$H(x,y,p_x,p_y) \;=\; \frac{\pi_x^2+\pi_y^2}{2M}, \quad \text{where} \quad \pi_x \equiv p_x + \frac{eB}{2c}y \quad \text{and} \quad \pi_y \equiv p_y - \frac{eB}{2c}x \,. \tag{6}$$

   (a) Write down the Hamilton-Jacobi equations for the Hamiltonian (H) and solve them.

   (b) Show that $x_0 \equiv x + \frac{c}{eB}\pi_y$ and $y_0 \equiv y - \frac{c}{eB}\pi_x$ are constants of the motion and explain their physical meaning.

   (c) Replace the classical quantities x, y, p_x and p_y with quantum operators obeying the canonical commutation relations and compute all the commutators between quantum analogues of π_x, π_y, x_0 and y_0.

   (d) Show that the non-hermitian operator $\hat{a} \equiv \sqrt{\frac{c}{2eB\hbar}}(\hat{\pi}_x + i\hat{\pi}_y)$ obeys $[\hat{a},\hat{a}^\dagger] = 1$.

   (e) Rewrite the quantum Hamiltonian $\hat{H} = \frac{1}{2M}(\hat{\pi}_x^2 + \hat{\pi}_y^2)$ in terms of $\hat{a}$ and $\hat{a}^\dagger$ and show that its spectrum consists of discrete Landau levels $E = E_0 + n\frac{eB\hbar}{Mc}$.

   (f) Show that the operators $\hat{x}_0$ and $\hat{y}_0$ are conserved, *i.e.*, commute with the Hamiltonian $\hat{H}$.

   (g) Use the commutation relations between $\hat{x}_0$ and $\hat{y}_0$ to show that each Landau level is infinitely degenerate.