1. This problem is about the Pauli matrices. The Pauli matrices are defined as

$$\sigma_x = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \sigma_y = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \sigma_z = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \quad (1)$$

Suppose a general 2×2 matrix $\mathbf{X}$ is written in terms of the Pauli matrices:

$$\mathbf{X} = a_0 \mathbf{I}_2 + \vec{\sigma} \cdot \vec{a} = a_0 \mathbf{I}_2 + \sigma_1 a_1 + \sigma_2 a_2 + \sigma_3 a_3 , \quad (2)$$

where $a_{0,1,2,3}$ are numbers and $\mathbf{I}_2$ is the 2×2 identity matrix.

(a) Write down the 2×2 matrix $\mathbf{X}$ explicitly.

(b) What are the commutator and anticommutator of two Pauli matrices, *i.e.* what are

$$[\sigma_i, \sigma_j] \text{ and } \{\sigma_i, \sigma_j\} \ ? \quad (3)$$

Justify your results.

(c) Write the product of two Pauli matrices, $\sigma_i \sigma_j$, as a sum of two terms, one with no Pauli matrices and one with one Pauli matrix.

(d) Evaluate the traces

$$\text{Tr}\left[\sigma^j\right] , \text{Tr}\left[\sigma^j \sigma^k\right] , \text{Tr}\left[\sigma^j \sigma^k \sigma^p\right] , \text{Tr}\left[\sigma^j \sigma^k \sigma^p \sigma^q\right] . \quad (4)$$

(e) How are $\text{Tr}[\mathbf{X}]$ and $\text{Tr}[\mathbf{X}\sigma^k]$ related to the $a_{0,1,2,3}$?

(f) From part (a) determine the $a_{0,1,2,3}$ in terms of the matrix elements X_{ij}.

2. This problem is about the polarization states of a silver atom. As discussed in class, silver atoms have only two independent polarizations, so in this problem we are dealing with a two-dimensional Hilbert space and its bases $(|Z+\rangle, |Z-\rangle)$, *etc.*

Stern-Gerlach experiments with multiple magnetic gaps tell us that

$$|\langle Z \pm | X \pm \rangle|^2 = |\langle Z \pm | Y \pm \rangle|^2 = |\langle X \pm | Y \pm \rangle|^2 = \frac{1}{2} , \quad (5)$$

where $|Z+\rangle$ is a normalized ket vector representing a quantum state in which the z-component of the atom's magnetic moment has definite value $\mu_z = +\mu_0 \equiv +e\hbar/2M_e c$. $|Z-\rangle$ represents the state that has $\mu_z = -\mu_0$ and same for the $|X\pm\rangle$ and $|Y\pm\rangle$.

(a) Use eq. (5) to show that after a physically-irrelevant change of the overall phases of the ket vectors $|Z\pm\rangle$, $|X\pm\rangle$ and $|Y\pm\rangle$, the six quantum states are related to each other via

$$|X\pm\rangle = \sqrt{\frac{1}{2}}|Z+\rangle \pm \sqrt{\frac{1}{2}}|Z-\rangle , \ |Y\pm\rangle = \sqrt{\frac{1}{2}}|Z+\rangle \pm i\sqrt{\frac{1}{2}}|Z-\rangle . \quad (6)$$

(b) Construct the operators $\hat{\mu}_x$, $\hat{\mu}_y$ and $\hat{\mu}_z$ for the three components of the atom's magnetic moment and use eqs. (6) to write down all the matrix elements of those operators in the $|Z\pm\rangle$ basis.

(c) For an arbitrary direction $\vec{n}$ ($\vec{n}^2 = 1$), the n-component of the atom's magnetic moment is $\mu_n = \vec{n} \cdot \hat{\mu}$. Write down the operator $\hat{\mu}_n = n_x \hat{\mu}_x + n_y \hat{\mu}_y + n_z \hat{\mu}_z$ and compute its eigenvalues. Explain the physical meaning of your result.

3. This problem is about coherent states. Consider the harmonic oscillator, $\hat{H} = \hbar\omega(\hat{a}^\dagger \hat{a} + \frac{1}{2})$. Most quantum mechanics texts describe how all interesting properties of this quantum system can be obtained algebraically from the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$. In particular, there is a quantum state $|0\rangle$ (the ground state of $\hat{H}$) that is annihilated by the operator $\hat{a}$. The same argument applied to the operator $\hat{a}' \equiv \hat{a} - \alpha$ and its Hermitian conjugate $\hat{a}'^\dagger \equiv \hat{a}^\dagger - \alpha^*$ (α being an arbitrary complex number) implies that there is a quantum state $|\alpha\rangle$ — called a *coherent state* — satisfying $\hat{a}' |a\rangle = 0$, i.e., $\hat{a} |\alpha\rangle = \alpha |\alpha\rangle$.

(a) Coherent states of the harmonic oscillator do not have definite energies, but for highly excited coherent states $\Delta E \ll E$. To demonstrate this fact, compute the expectation values of $\hat{n} = \hat{a}^\dagger \hat{a}$ and $\hat{n}^2$ in a coherent state $|\alpha\rangle$ and show that $\Delta n = \sqrt{\langle \hat{n} \rangle}$.

(b) Compute $\langle n|\alpha\rangle$ and use the result to show that $|\alpha\rangle \propto \exp(\alpha\hat{a}^\dagger)\,|0\rangle$. *Hint*: Note that $|n\rangle = [\hat{a}^\dagger]^n/\sqrt{n!}\,|0\rangle$.

(c) Show that the definition

$$|\alpha\rangle \;=\; e^{-|\alpha|^2/2} \exp(\alpha\hat{a}^\dagger)\,|0\rangle \tag{7}$$

produces normalized coherent states and compute the overlap $|\langle\alpha|\beta\rangle|^2$.

(d) Show that the set of all coherent states is an overcomplete basis of the oscillator's Hilbert space. Specifically,

$$\int_{\mathbf{C}} \frac{d^2\alpha}{\pi}\, |\alpha\rangle\langle\alpha| \;=\; \mathbf{1} \tag{8}$$

where $d^2\alpha \equiv d\mathrm{Re}(\alpha)\, d\mathrm{Im}(\alpha)$, and $\mathbf{C}$ implies that the integral is over the complex α plane.