944 QUANTUM MECHANICS - Mid-Term Exam Due: Friday, 31 October, 2008

No collaboration is permitted on the mid-term exam. You may freely use the literature, but with diligent referencing. Do not include rough notes or programming efforts; give only your final logical development in legible handwriting. Presentation will be a primary factor in grading.

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1. This problem is about time-dependent perturbation theory and its relation with time-independent perturbation theory.

   (a) When the potential V is time-independent, work out $\langle s|\tilde{T}(t,0)|s\rangle$ to second order and identify $\Delta^{(1)}$, $\Delta^{(2)}$ and the "wave-function renormalization" Z_i in the expansion of

   $$\begin{aligned}\langle s|\tilde{T}(t,0)|s\rangle &= Z_i\, e^{-i\Delta Et/\hbar} \;+\; \textit{rapidly oscillating terms} \\ &= Z_i \;-\; \frac{i}{\hbar}\left(\Delta_i^{(1)}+\Delta_i^{(2)}\right)t \;+\; \frac{1}{2!}\left(-\frac{i}{\hbar}\Delta_i^{(1)}t\right)^2 \;+\; \vartheta(V^3) \quad (1)\end{aligned}$$

   and show that they agree with the results from time-independent perturbation theory, Eqs. (5.1.42), (5.1.44) and (5.1.48b) in Sakurai. Note that this identification is done in the $t\to\infty$ limit where rapidly oscillating terms are dropped. Explain why this identification works.

   (b) Now consider a harmonic perturbation $V = V_0\cos\omega t$. Work out the second-order energy shift. Does your expression recover the result from time-independent perturbation theory in the limit $\omega\to 0$? Explain your answer.

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2. This problem is about the variational method for a non-linear Schrödinger equation, which can be written as

   $$\left[-\frac{\hbar^2}{2m}\nabla_i\nabla_i \;+\; g|\psi(r_i)|^2 \;+\; V(r_i)\right]\psi(r_i) \;=\; \epsilon\psi(r_i)\;, \qquad (2)$$

   where ϵ is the eigenvalue and $V(r_i)=V(r)=\frac{1}{2}m\omega^2r^2$ is the potential for the harmonic oscillator. Notice that the equation is non-linear because the wavefunction appears inside the square brackets.

   (a) What is the physical significance of the non-linearity? Give a specific example of a physical system that is described by a non-linear Schrödinger equation. This will probably require you to do a little research.

   (b) Taking a Gaussian as your variational ansatz, miminimize the total energy subject to the constraint that the wavefunction be normalized; i.e.

   $$\int d^3r|\psi(r_i)|^2 \;=\; 1\;. \qquad (3)$$

   For this part you may take the problem to be one-dimensional; *i.e.* $r_i\to x$. Note that $E\neq\epsilon$, due to the non-linearity.

   HINT: To find the correct object to minimize, find the functions I and F (constructed in class), which when minimized, give the non-linear Schrödinger equation (via the Euler-Lagrange equations).

(c) Next treat this problem in three dimensions. What happens for positive vs. negative coupling g? Find the critical point for which the solution collapses to a singularity.

(d) Show that for (b) and (c) you obtain the correct result in the limit $g \to 0$. Is perturbation theory useful in this case?