944 QUANTUM MECHANICS - Homework #6 Due: Friday, 12 December, 2008

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1. This problem is about the Bogoliubov transformation. A common tool in studying many-body quantum systems is the operator transform. Suppose the particle creation and annihilation operators $a_i^\dagger$ and a_i can be algebraically expressed in terms of a new set of operators $b_i^\dagger$ and b_i that obey the same canonical commutation relations:

$$[b_i, b_j] \;=\; [b_i^\dagger, b_j^\dagger] \;=\; 0 \qquad [b_i, b_j^\dagger] \;=\; \delta_{ij} \; . \tag{1}$$

The operators $b_i^\dagger$ and b_i are often said to create/annihilate quasiparticles. The commutation relations, Eq. (1), imply that there is a unique state $|B\rangle$ that is annihilated by all b_i; this state is usually referred to as the quasiparticle vacuum, the states of the form $b_i^\dagger |B\rangle$ are the one-quasiparticle states, *etc.* Whenever the quasiparticles can be labeled by the same quantum numbers (e.g. $\vec{k}$) as the original bosonic particles of the theory, it is often convenient to make a unitary operator transform:

$$b_i \;=\; U a_i U^\dagger, \qquad b_i^\dagger \;=\; U a_i^\dagger U^\dagger \; , \tag{2}$$

where U is a unitary operator in the Fock space, usually of the form $\exp(X)$ for some anti-hermitian polynomial X in a_i and $a_i^\dagger$.

(a) Show that the unitarity of U automatically guarantees that b_n and $b_n^\dagger$ satisfy Eq. (1), and that the quasiparticle state $|B\rangle = U|0\rangle$ is the quasiparticle vacuum.

(b) Verify that for $X = \sum_n (c_n a_n^\dagger - c_n^* a_n)$, $\exp(X) a_n \exp(-X) = a_n - c_n$. This transform is a c-number shift.

(c) Now let $X = \sum_n \frac{1}{2}\eta_n (e^{i\lambda_n}(a_n^\dagger)^2 - e^{-i\lambda_n}(a_n)^2)$ (η_n and λ_n are real). Show that for this $U = \exp(X)$, Eqs. (2) define a diagonal canonical transform:

$$b_i \;=\; a_i \cosh \eta_i \;-\; e^{i\lambda_i} a_i^\dagger \sinh \eta_i, \qquad b_i^\dagger \;=\; \cosh \eta_i a_i^\dagger \;-\; e^{-i\lambda_i} \sinh \eta_i a_i \; . \tag{3}$$

(d) In order to see the utility of the Bogoliubov transformation, consider the simple case of one creation/annihilation operator pair with $\lambda = \pi$. We then have

$$b \;=\; a \cosh \eta \;+\; a^\dagger \sinh \eta \; . \tag{4}$$

Use this transformation to obtain the eigenvalues of the following Hamiltonian:

$$H \;=\; \hbar\omega a^\dagger a \;+\; \frac{1}{2} V (aa + a^\dagger a^\dagger) \; . \tag{5}$$

Also give the upper limit on V for which this can be done.

(e) Write down the ground state of the Hamiltonian above in terms of the number states $a^\dagger a |n\rangle = n |n\rangle$.