944 QUANTUM MECHANICS - <u>Final Exam</u> Due: Noon, Tuesday, 23 December, 2008

No collaboration is permitted on the final exam. You may freely use the literature, but with diligent referencing. Do not include rough notes or programming efforts; give only your final logical development in legible handwriting. Presentation will be a primary factor in grading.

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1. This problem is about time-independent perturbation theory for a two-particle system in a finite volume. Consider a three-dimensional cubic volume with sides each of length L. Particles are placed inside this volume subject to periodic boundary conditions.

   (a) What are the eigenstates of a single particle confined to this volume?

   (b) What are the eigenstates of two non-interacting particles confined to this volume with zero total momentum?

   (c) Show that if $V(r)$ is a two-particle interaction that depends only on the distance r between the particles, the matrix element of the interaction in momentum space may be reduced to

$$\left\langle k_i^3 k_i^4 \right| V \left| k_i^1 k_i^2 \right\rangle \;=\; \delta^3\left(k_i^1 + k_i^2 - k_i^3 - k_i^4\right) \frac{1}{(2\pi)^3} \int d^3r V(r) e^{-iq_i r_i} \;, \qquad (1)$$

   where $q_i \equiv k_i^3 - k_i^1$ is the momentum transfer.

   (d) What is the ground state energy of two particles confined to the cubic volume that interact with each other via an interaction

$$V \;=\; \eta\, \delta^3\left(r_i^1 - r_i^2\right) \;, \qquad (2)$$

   out to second order in perturbation theory, assuming that $\eta/L \ll 1$? Do not evaluate any sums or integrals that may occur.

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2. This problem is about scattering in a finite volume. Consider the very low-energy scattering of two particles interacting via the potential defined above.

   (a) From the Lippmann-Schwinger equation determine the scattering amplitude out to second order in the potential, and thereby show that

$$\eta \;=\; -\frac{4\pi a}{M}\left[1 \;-\; 4\pi a \int \frac{d^3p_i}{(2\pi)^3} \frac{1}{p_i p_i + i\epsilon} + \ldots\right] \;, \qquad (3)$$

   where a is the scattering length.

   (b) Use this result to show that the energy shift of two particles in the cubic volume is

$$\Delta E_0 \;=\; -\frac{4\pi a}{ML^3}\left[1 \;+\; \left(\frac{a}{\pi L}\right)\left(\sum_{n_i \neq 0}^{\Lambda_j} \frac{1}{n_i n_i} - 4\pi\Lambda_j\right) + \ldots\right] \;, \qquad (4)$$

   where it is understood that the limit $\Lambda_j \to \infty$ is taken, and the sum extends over all integer triplets up to a cutoff Λ_j.