944 QUANTUM MECHANICS - Homework #6. Due: Friday, November 30, 2007.

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1. This problem is about the scattering of slow particles from a spherically symmetric compact potential, $V(r)$. *Compact* means that $V(r)$ becomes negligibly small outside of some finite radius b, and the particles are regarded as *slow* when $kb \ll 1$. An exact expression for the phase shift for a given partial wave ℓ is given by:

$$\frac{\sin\delta_\ell}{k} = -\frac{2m}{\hbar^2}\int_0^\infty j_\ell(kr)\, V(r)\, R_{k,\ell}(r)\, r^2\, \mathrm{d}r \,, \tag{1}$$

where $R_{k,\ell}(r)$ is a solution of the radial Schrödinger equation.

(a) Under certain circumstances, at low incident energy we can approximate

$$R_{k,\ell}(r) \sim A\,(kr)^\ell \,, \tag{2}$$

near the origin. A is some constant. Using the fact that $j_\ell(kr) \sim (kr)^\ell$ near the origin, establish the *"rule of thumb"* that at low energies,

$$\delta_\ell \sim (kb)^{2\ell+1} \,. \tag{3}$$

What conclusions can you draw concerning the phase shifts for very low energy incident particles? How do the phase shifts relate to the differential scattering cross sections? What does this mean in terms of the angular dependence of the scattering cross section, at low energies? Discuss under what conditions Eq. (2) is true.

(b) For $\ell = 0$ and very low energy (Hint: just set the energy to zero), show that the solution of the radial Schrödinger equation for $u(r) = r\,R(r)$ is given by:

$$u(r) = r - a_s \,, \tag{4}$$

where a_s is called the *scattering length*. Relate the scattering length a_s to the s-wave phase shift δ_0 and the scattering cross section σ.

(c) Compute the scattering length for a repulsive spherical square well $V(r < b) = V_0 > 0$, $V(r > b) = 0$ and for an attractive spherical square well (same, but $V_0 < 0$). Show that a barrier has $0 < a_s < b$ while a well that is too shallow to have bound states has $a_s < 0$.

(d) What happens when a well has a bound state that is barely bound, *i.e.*, its energy ϵ is negative but $|\epsilon| \ll |V_0|$, or is about to acquire a new bound state given a minor increase in its depth or width?

(e) When the scattering length a_s becomes very large, the approximation $|\delta_0| \ll 1$ may break down; for a well with a barely-bound energy level $-\epsilon$ (or a *virtual*, almost-bound level $+\epsilon$), this happens when $E = O(|\epsilon|)$. Prove this. For such wells compute the total scattering cross section and its energy dependence.

2. This problem is about the Born approximation (Merzbacher, chapter 13, excercise 13.11). If

$$V(r) = \frac{C}{r^n} , \tag{5}$$

obtain the functional dependence of the Born scattering amplitude on the scattering angle. Discuss the reasonableness of the result qualitatively. What values of n give a meaningful answer?