**944 QUANTUM MECHANICS** - Homework #3. Due: Friday, October 19, 2007.

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1. This problem is about the Wigner-Eckart theorem, which here we will write as,

$$\langle j', m', \alpha' | \hat{T}^{(\ell)}_{\mu} | j, m, \alpha \rangle \;=\; \langle j', \alpha' || \hat{T}^{(\ell)} || j, \alpha \rangle \cdot \langle \ell, j; j', m' | \ell, j; \mu, m \rangle \tag{1}$$

and its application to matrix elements of vector operators.

(a) Use the Wigner-Eckart theorem to show that for $j' = j$ and for any vector operator $\hat{A}_i$,

$$\langle j, m', \alpha' | \hat{A}_i | j, m, \alpha \rangle \;=\; \langle j, \alpha' || \hat{A}^{(1)} || j, \alpha \rangle \cdot \frac{\langle j, m' | \hat{J}_i | j, m \rangle}{\langle j || \hat{J}^{(1)} || j \rangle} \, . \tag{2}$$

(b) Show that the matrix elements of the scalar product $\hat{A}_i \hat{J}_i$ are related to the reduced matrix elements of $\hat{A}^{(1)}$ via

$$\langle j, m, \alpha' | \hat{A}_i \hat{J}_i | j, m, \alpha \rangle \;=\; C_j \langle j, \alpha' || \hat{A}^{(1)} || j, \alpha \rangle \, , \tag{3}$$

where the coefficient $C_j$ depends only on $j$ and is the same for all vector operators $\hat{A}_i$ and all rotationally-invariant quantum numbers $\alpha, \alpha'$.

(c) Combine eqs. (2) and (3) and the fact that $\hat{J}_i$ is itself a vector operator to prove that

$$\langle j, m', \alpha' | \hat{A}_i | j, m, \alpha \rangle \;=\; \frac{1}{\hbar^2 j(j+1)} \langle j, m, \alpha' | \hat{A}_j \hat{J}_j | j, m, \alpha \rangle \cdot \langle j, m' | \hat{J}_i | j, m \rangle \, . \tag{4}$$

This formula is often called *the projection theorem.*

(d) The magnetic moment of an electron is $\hat{\mu}_i = \frac{-e}{2M_e c}(\hat{L}_i + \hbar \hat{\sigma}_i)$. Hence, the magnetic moment of an atom is $\hat{\mu}_i = \frac{-e}{2M_e c}(\hat{L}_i + 2\hat{S}_i)$, where $\hat{L}_i$ is the combined orbital angular momentum of the atom's electrons and $\hat{S}_i$ is their combined spin.
The ground states of atoms usually have well-defined values of $\ell_{tot}$ and $s_{tot}$ as well as $j_{tot}$ and $m_j$ (the so-called LS coupling). Show that the matrix elements of the magnetic moment between such states are:

$$\langle \ell, s, j, m', \mathrm{rad} | \hat{\mu}_i | \ell, s, j, m, \mathrm{rad} \rangle \;=\; \frac{-e}{2M_e c} \, g \, \langle j, m' | \hat{J}_i | j, m \rangle \, , \tag{5}$$

(Lande's formula) and compute the *gyromagnetic factor* $g$ in terms of $\ell$, $s$ and $j$.