944 QUANTUM MECHANICS - Homework #2. Due: Friday, October 5, 2007.

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1. This problem is about the 2-dimensional isotropic harmonic oscillator. The Hamiltonian is

$$\hat{H} \;=\; (\hat{p}_1^2 + \hat{p}_2^2)/2m \;+\; k(\hat{q}_1^2 + \hat{q}_2^2)/2 \;, \tag{1}$$

where the p's and q's obey the usual commutation relations.

   (a) Construct $\hat{L}_3$ and verify that $[\hat{H}, \hat{L}_3] = 0$.

   (b) Consider the operators:

$$\begin{aligned}
\hat{F}_1 &= \frac{1}{2}\left(\frac{1}{\sqrt{mk}}\,\hat{p}_1\hat{p}_2 \;+\; \sqrt{mk}\,\hat{q}_1\hat{q}_2\right) \\
\hat{F}_2 &= \frac{1}{2}\left(\hat{q}_1\hat{p}_2 \;-\; \hat{q}_2\hat{p}_1\right) \\
\hat{F}_3 &= \frac{1}{4}\left(\frac{1}{\sqrt{mk}}\,(\hat{p}_1^2 - \hat{p}_2^2) \;+\; \sqrt{mk}\,(\hat{q}_1^2 - \hat{q}_2^2)\right) \;.
\end{aligned} \tag{2}$$

   Verify that $[\hat{H}, \hat{F}_i] = 0$. What is the physical significance of these operators? How is $\hat{F}_2$ related to $\hat{L}_3$?

   (c) Verify that the F_i satisfy the algebra of $SO(3)$:

$$\left[\;\hat{F}_i, \hat{F}_j\;\right] \;=\; i\hbar\epsilon_{ijk}\hat{F}_k \;. \tag{3}$$

   (d) How can this be, given that the harmonic oscillator lives in 2 dimensions??

2. This problem is about Schwinger's oscillator model of angular momentum, which makes use of the observation made in the previous problem. The $SO(3)$ rotation group has both single-valued and double-valued representations, corresponding to integral and half-integral values of j, respectively. Both kinds of representations become single valued in terms of the Spin(3) group (the double cover of $SO(3)$); Spin(3) is isomorphic to $SU(2)$. The $SU(2)$ picture of the spin group is more convenient for deriving the explicit rotation matrices, $\mathcal{D}^{(j)}_{m,m'}(\phi, \vec{n})$ for all representations (j). In this problem, we will construct the $\mathcal{D}^{(j)}_{m,m'}$ matrix elements as explicit polynomials of the matrix elements $U_{\alpha\beta}$ of the $SU(2)$ matrix $U(\alpha, \vec{n}) \equiv \exp(-i\frac{\alpha}{2}\vec{n}\cdot\vec{\sigma})$.

   Our starting point is a system of two independent harmonic oscillators whose creation and annihilation operators $\hat{a}_+^\dagger$, $\hat{a}_-^\dagger$,$\hat{a}_+$,$\hat{a}_-$ obey the canonical commutation relations

$$[\;\hat{a}_\alpha, \hat{a}_\beta\;] \;=\; 0 \;=\left[\;\hat{a}_\alpha^\dagger, \hat{a}_\beta^\dagger\;\right] \;,\quad \left[\;\hat{a}_\alpha, \hat{a}_\beta^\dagger\;\right] \;=\; \delta_{\alpha\beta} \;,\qquad \alpha, \beta \;=\; +, - \tag{4}$$

   and a trio of model angular momentum operators

$$\hat{J}_i \;=\; \frac{\hbar}{2} \sum_{\alpha,\beta} \sigma_{i,\alpha\beta}\; \hat{a}_\alpha^\dagger \hat{a}_\beta \;, \tag{5}$$

   where $\sigma_{i,\alpha\beta}$ are matrix elements of the Pauli matrices σ_i.

(a) Compute the commutators $\left[\ \hat{J}_i, \hat{a}_\alpha\ \right]$ and $\left[\ \hat{J}_i, \hat{a}^\dagger_\alpha\ \right]$.

(b) Verify that $\left[\ \hat{J}_i, \hat{J}_j\ \right]\ =\ i\hbar\epsilon_{ijk}\hat{J}_k$; it is this relation that allows us to treat the $\hat{J}_i$ as model angular momenta. (Hint: you've already shown this in problem 1.)

(c) Prove that

$$\hat{J}_i\hat{J}_i\ =\ \hbar^2\frac{\hat{N}}{2}\left(\frac{\hat{N}}{2}\ +\ 1\right)\ ,\qquad \text{where}\quad \hat{N}\ \equiv\ \hat{a}^\dagger_+\hat{a}_+\ +\ \hat{a}^\dagger_-\hat{a}_-\ . \tag{6}$$

(Hint: first express $\hat{J}_z$ and $\hat{J}_\pm$ explicitly in terms of $\hat{a}_\pm$ and $\hat{a}^\dagger_\pm$; then compute $\hat{J}_i\hat{J}_i = \hat{J}_z^2 + \frac{1}{2}\{\hat{J}_+, \hat{J}_-\}$.)

(d) Show that for this model the states with definite values of j and m are precisely the states with definite numbers of oscillatorial quanta n_+ and n_-. Specifically,

$$|j, m\rangle = |n_+ = j+m, n_- = j-m\rangle = ((j+m)!\,(j-m)!)^{-1/2}\left(\hat{a}^\dagger_+\right)^{j+m}\left(\hat{a}^\dagger_-\right)^{j-m}|0\rangle \tag{7}$$

where $|0\rangle$ is the ground state of the two-oscillator system.

(e) Now suppose that for some unitary operator $\hat{V}$,

$$\hat{V}\,|0\rangle\ =\ |0\rangle\qquad\text{and}\qquad \hat{V}\hat{a}^\dagger_\alpha\hat{V}^\dagger\ =\ \sum_\beta \hat{a}^\dagger_\beta U_{\beta\alpha} \tag{8}$$

where $U_{\beta\alpha}$ is an $SU(2)$ matrix. Show that the relations, eq. (8), inevitably lead to

$$\hat{V}\,|j, m\rangle\ =\ \sum_{m'} |j, m'\rangle\,\mathcal{D}^{(j)}_{m',m} \tag{9}$$

and compute the matrix elements $\mathcal{D}^{(j)}_{m',m}$ as polynomials of the matrix elements of U.

Notice that for $j = 1/2$ the $\mathcal{D}^{(1/2)}$ matrix is U. Therefore, this exercise gives us the $\mathcal{D}^{(j)}$ matrices for states of all angular momenta j in terms of the two-by-two matrix for the states of $j = 1/2$.

(f) Prove the following lemma: For any operator $\hat{B}$ and a finite set of operators $\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_N$ satisfying the commutation relations $\left[\hat{A}_n, \hat{B}\right] = \sum_{n'} \hat{A}_{n'}C_{n',n}$ where $C_{n',n}$ is a finite N-by-N matrix of c-numbers,

$$\exp(t\hat{B})\hat{A}_n\exp(-t\hat{B})\ =\ \sum_{n'} \hat{A}_{n'}\left[\exp(tC)\right]_{n',n}\ . \tag{10}$$

(Hint: differentiate the left hand side of eq. (10) with respect to t and then solve the differential equation.)

(g) Now consider the rotation operators $\hat{R}(\phi, \vec{n})\ =\ \exp(\frac{\phi}{i\hbar}n_i\hat{J}_i)$ generated by the angular momentum operators, eq. (5). Show that these rotation operators do satisfy eq. (8), and the $SU(2)$ matrix $U_{\alpha\beta}$ happens to be $\left[\exp(\frac{\phi}{2i\hbar}n_i\hat{\sigma}_i)\right]_{\beta\alpha}$.

(h) Finally, explain why the $\mathcal{D}^{(j)}$ matrices that you've computed in this problem would work for any physical system with a well-defined angular momentum, and not just for the Schwinger model of this problem.