944 QUANTUM MECHANICS - Homework #1. Due: Friday, September 21, 2007.

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1. This problem is about the orbital angular momentum operator, $\hat{L}_i = \epsilon_{ijk}\hat{X}_j\hat{P}_k$.

   (a) Using canonical commutation relations between components of $\hat{X}_i$ and $\hat{P}_i$, show that

   $$\left[\ \hat{X}_i, \hat{L}_j\ \right] \ = \ i\hbar\epsilon_{ijk}\hat{X}_k \quad \text{and} \quad \left[\ \hat{P}_i, \hat{L}_j\ \right] \ = \ i\hbar\epsilon_{ijk}\hat{P}_k \ . \tag{1}$$

   (b) Show that

   $$\left[\ \hat{L}_i, \hat{L}_j\ \right] \ = \ i\hbar\epsilon_{ijk}\hat{L}_k \quad \text{and} \quad \epsilon_{ijk}\hat{L}_j\hat{L}_k \ = \ i\hbar\hat{L}_i \ . \tag{2}$$

   (c) Define $\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$ and show that

   $$\left[\ \hat{L}_z, \hat{L}_\pm\ \right] \ = \ \pm\hbar\hat{L}_\pm \quad \text{and} \quad \left[\ \hat{L}_+, \hat{L}_-\ \right] \ = \ 2\hbar\hat{L}_z \ . \tag{3}$$

   (d) Show that in the spherical coordinate basis

   $$\begin{aligned} \hat{L}_z \ \Psi(r,\theta,\phi) \ &= \ -i\hbar\frac{\partial}{\partial\phi} \ \Psi(r,\theta,\phi) \ , \\ \hat{L}_\pm \ \Psi(r,\theta,\phi) \ &= \ -i\hbar \ e^{\pm i\phi}\left(\pm i\frac{\partial}{\partial\theta} \ - \ \frac{1}{\tan\theta}\frac{\partial}{\partial\phi} \right) \ \Psi(r,\theta,\phi) \ . \end{aligned} \tag{4}$$

   (e) Compute $\hat{L}_i\hat{L}_i \ \Psi(r,\theta,\phi)$ in the same basis.
   (Hint: use $\hat{L}_i\hat{L}_i = \hat{L}_z^2 + \frac{1}{2}\hat{L}_+\hat{L}_- + \frac{1}{2}\hat{L}_-\hat{L}_+$.)

2. This problem is about rotations in three dimensions. For a rotation by angle α around an axis of a unit vector $\vec{n}$, the rotation matrix is

   $$R_{ij}(\alpha, \vec{n}) \ = \ \delta_{ij}\cos\alpha \ - \ \epsilon_{ijk}n_k\sin\alpha \ + \ n_in_j(1-\cos\alpha) \ . \tag{5}$$

   A product of two rotations $R(\alpha', \vec{n}') \cdot R(\alpha'', \vec{n}'')$ is itself a rotation by some angle α around some axis $\vec{n}$. To determine α and $\vec{n}$, we demand

   $$R_{ij}(\alpha, \vec{n}) \ = \ R_{ik}(\alpha', \vec{n}') \cdot R_{jk}(\alpha'', \vec{n}'') \ , \tag{6}$$

   and then substitute eq. (5) and solve for α and $\vec{n}$.

   (a) Show that to second order in α' and α'',

   $$\alpha\vec{n} \ \simeq \ \alpha'\vec{n}' \ + \ \alpha''\vec{n}'' \ + \ \frac{1}{2}\alpha'\alpha''\vec{n}' \times \vec{n}'' \ . \tag{7}$$

   (b) Prove $trR \equiv R_{ii} = 1 + 2\cos\alpha$ and $\epsilon_{ijk}R_{jk} = -2n_i\sin\alpha$ and use these formulas to derive exact expressions for $\cos\alpha$ and $\vec{n}\sin\alpha$ in terms of α', α'', $\vec{n}'$ and $\vec{n}''$.

3. This problem is about Pauli matrices and the relation between abstract rotations in spin space and rotations in three dimensions.

(a) Show that for any two vectors $\vec{a}$ and $\vec{b}$,

$$(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = (\vec{a}\cdot\vec{b})\mathbf{1} \ + \ i(\vec{a}\times\vec{b})\cdot\vec{\sigma}. \tag{8}$$

(b) Show that for any angle α and for any unit vector $\vec{n}$,

$$U(\alpha,\vec{n}) \equiv \exp(-i\frac{\alpha}{2}\vec{n}\cdot\vec{\sigma}) = \cos(\frac{\alpha}{2})\mathbf{1} - i\sin(\frac{\alpha}{2})\ \vec{n}\cdot\vec{\sigma}\ . \tag{9}$$

(c) Solve for α and $\vec{n}$ such that

$$U(\alpha,\vec{n}) \ = \ U(\alpha',\vec{n}')\ U(\alpha'',\vec{n}'')\ . \tag{10}$$

(d) The commutation relations between generators of a Lie algebra completely determine the products of their exponentials. Consider a vector of 2×2 hermitian matrices, $\vec{S} \equiv \frac{\hbar}{2}\vec{\sigma}$; commutation relations between its compontents are exactly the same as between the components $\hat{J}_i$ of the angular momentum operator. Since the definition of the U matrices amounts to $U(\alpha,\vec{n}) = \exp(\alpha\vec{n}\cdot\vec{S}/i\hbar)$, which is precisely analogous to $\hat{R}(\alpha,\vec{n}) = \exp(\alpha\vec{n}\cdot\vec{\hat{J}}/i\hbar)$, we expect the product rule for the U matrices to be exactly the same as for the unitary rotation operators $\hat{R}$, which in turn sould be indentical to the product rule for the R_{ij} rotation matrices given in eq. (6).

Verify that a solution of eq. (10) is automatically a solution of eq. (6).

Is the converse also true?