961 QUANTUM FIELD THEORY - Final Exam - Due: 5/10/2007.

1. **Linear Sigma Model**.

   Problem 4.3, Peskin and Schroeder

2. **Regularization Schemes**.

   Compute the one-loop S-channel scattering diagram in $\lambda\phi^4$ theory (which we evaluated in class using dimensional regularization) using a sharp cutoff and using Pauli-Villars regularization and show explicitly how the cutoff parameters of the two methods are related.

3. **Muon Decay**.

   Consider the muon decay, $\mu^- \to e^-\bar{\nu}_e\nu_\mu$. Since neutrinos are hard to detect experimentally, the readily measurable quantities for this process are the total muon decay rate $\Gamma_\mu = 1/\tau_\mu$ and the energy distribution of electrons produced by decaying muons; this exercise will consist of calculating these quantities from the Fermi theory of weak interactions. According to the Fermi theory, the matrix element for muon decay is

   $$\left\langle e^-, \bar{\nu}_e, \nu_\mu \right| \mathcal{M} \left| \mu^- \right\rangle = \frac{G_F}{\sqrt{2}} \left[\bar{u}(\nu_\mu)(1-\gamma^5)\gamma^\alpha u(\mu^-)\right] \times \left[\bar{u}(e^-)(1-\gamma^5)\gamma_\alpha v(\bar{\nu}_e)\right]. \quad (1)$$

   The Standard Model of particle interactions produces essentially the same answer at tree level in perturbation theory.

   (a) Sum the absolute square of the amplitude over the final particle spins and average over the initial muon's spin. Show that altogether,

   $$\frac{1}{2}\sum_{\substack{\text{all}\\ \text{spins}}} \left|\left\langle e^-, \bar{\nu}_e, \nu_\mu \right| \mathcal{M} \left| \mu^- \right\rangle\right|^2 = 64\, G_F^2\, (p_\mu \cdot p_{\bar{\nu}})\,(p_e \cdot p_\nu). \quad (2)$$

   (b) Consider a generic three-body decay of some particles of mass M_0 into three particles of respective masses m_1, m_2, and m_3. Show that in the rest frame of the original particle the partial decay rate is given by

   $$d\Gamma = \frac{1}{2M_0} \times \overline{|\mathcal{M}|^2} \times \frac{d^3\Omega}{256\pi^5} \times dE_1\, dE_2\, dE_3\, \delta(E_1+E_2+E_3-M_0)\ , \quad (3)$$

   where the bar indicates an average and where $d^3\Omega$ refers to three angular variables parameterizing the directions of the three final-state particles relative to some external frame but not affecting the angles between the three momenta. For example, one may use two angles to describe the orientation of the decay plane (the three momenta are coplanar, $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = 0$) and one more angle to fix the direction of *e.g.* $\mathbf{p}_1$ in that plane. Altogether, $\int d^3\Omega = 4\pi \times 2\pi = 8\pi^2$.

   (c) Show that when $m_1 = m_2 = m_3 = 0$, the kinematically allowed range of the final particles' energies is given by

   $$0 \le E_1, E_2, E_3 \le \frac{1}{2}M_0 \qquad \text{while} \quad E_1+E_2+E_3 = M_0\ . \quad (4)$$

   (For non-zero masses $m_{1,2,3}$ this range is much more complicated.)

(d) The electron and the neutrinos are much lighter then the muon, so in most decay events all three final-state particles are ultra-relativistic. This allows us to approximate $m_e \approx m_\nu \approx m_{\bar{\nu}} \approx 0$, which greatly simplifies the last part of this exercise: Integrate the muon's partial decay rate over the final particle energies and derive first $d\Gamma/dE_e$ and then the total decay rate.