Now insert $TT = \mathbb{1}$ everywhere and apply rules:

$$T\psi(t,x_i)T = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \Big(a^{-s}_{-p_i} [u^s(p)]^* e^{ip\cdot x}$$

$$+ b^{-s}_{-p_i} [v^s(p)]^* e^{-ip\cdot x} \Big)$$

Action of T on operator

Action of T on c-numbers

Now change variables $p \to \hat{p}$ and use $[u]^*$, $[v]^*$ relations:

$$T\psi(t,x_i)T = (-\gamma_1\gamma_3) \int \frac{d^3\hat{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\hat{p}}}} \sum_s \Big(a^{-s}_{\hat{p}} u^{-s}(\hat{p}) e^{i\hat{p}\cdot(t,-x_i)}$$

$$+ b^{-s\,\dagger}_{\hat{p}} v^{-s}(\hat{p}) e^{-i\hat{p}\cdot(t,-x_i)}$$

$\Big\{$ Use $\hat{p}\cdot(t,-x_i) = (p^0,-p_i)\cdot(t,-x_i) = -\hat{p}(-t,x_i)$ $\Big\}$

$$\Rightarrow \boxed{T\psi(x_i,t)T = (-\gamma_1\gamma_3)\psi(-t,x_i)}$$

Simple transformation on ψ !!

$$T\bar{\psi}T = (T\psi T)^\dagger (\gamma^0)^* = \psi^\dagger(-t,x_i)[-\gamma_1\gamma_3]^\dagger \gamma^0$$

$$\Rightarrow \boxed{T\bar{\psi}(x_i,t)T = \bar{\psi}(-t,x_i)\gamma_1\gamma_3}$$

Now check action of T on various bilinears.

SCALAR

$$T\bar{\psi}\psi(t,x_i)T = \bar{\psi}\gamma^1\gamma^3(-\gamma^1\gamma^3)\psi(-t,x_i)$$
$$= +\bar{\psi}\psi(-t,x_i)$$

PSEUDOSCALAR

$$T\,i\bar{\psi}\gamma^5\psi\,T = -i\,\bar{\psi}(\gamma^1\gamma^3)\gamma^5(-\gamma^1\gamma^3)\psi(-t,x_i)$$
$$= -i\,\bar{\psi}\gamma^5\psi(-t,x_i)$$

VECTOR

$$T\bar{\psi}\gamma^\mu\psi T = \bar{\psi}\gamma^1\gamma^3(\gamma^\mu)^*(-\gamma^1\gamma^3)\psi(-t,x_i)$$
$$= \begin{cases} +\bar{\psi}\gamma^\mu\psi(-t,x_i) & \mu=0 \\ -\bar{\psi}\gamma^\mu\psi(-t,x_i) & \mu=i \end{cases}$$

AXIAL VECTOR

SAME AS VECTOR!

Charge Conjugation

Define:

$$\boxed{C\, a_p^s\, C = b_p^s \quad , \quad C\, b_p^s\, C = a_p^s}$$

takes <u>fermion</u> into <u>antifermion</u> w/ same spin

& " <u>antifermion</u> " <u>fermion</u> w/ " "

What is action of C on ψ ??

Recall:

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\, \xi^s \\ \sqrt{p\cdot\bar\sigma}\, \xi^s \end{pmatrix} , \quad v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\, \xi^{-s} \\ -\sqrt{p\cdot\bar\sigma}\, \xi^{-s} \end{pmatrix}$$

$$\xi^{-s} = -i\sigma^2 (\xi^s)^*$$

$$\sqrt{p\cdot\sigma}\, \sigma^2 = \sigma^2 \sqrt{p\cdot\bar\sigma^*}$$

$$\sqrt{p\cdot\bar\sigma}\, \sigma^2 = \sigma^2 \sqrt{p\cdot\sigma^*}$$

$$(v^s(p))^* = \begin{pmatrix} \sqrt{p\cdot\sigma}\, (-i\sigma^2(\xi^s)^*) \\ -\sqrt{p\cdot\bar\sigma}\, (-i\sigma^2(\xi^s)^*) \end{pmatrix}^*$$

$$= \begin{pmatrix} -i\sigma^2 \sqrt{p\cdot\bar\sigma^*}\, \xi^{s*} \\ i\sigma^2 \sqrt{p\cdot\sigma^*}\, \xi^{s*} \end{pmatrix}^* = \begin{pmatrix} -i\sigma^2 \sqrt{p\cdot\bar\sigma}\, \xi^s \\ i\sigma^2 \sqrt{p\cdot\sigma}\, \xi^s \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\sigma}\, \xi^s \\ \sqrt{p\cdot\bar\sigma}\, \xi^s \end{pmatrix}$$

$$\Rightarrow \quad (v^s(p))^* = -i\gamma^2 u^s(p)$$

$$\Rightarrow \quad \boxed{u^s(p) = -i\gamma^2 (v^s(p))^*}$$

Similarly,

$$\boxed{v^s(p) = -i\gamma^2 (u^s(p))^*}$$

(simply c.c. of above!!)

we're ready!

$$C\psi(x)C = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}$$

$$\times \sum_s \left(-i\gamma^2 (v^s(p))^* b_p^s e^{-ip\cdot x} - i\gamma^2 (u^s(p))^* a_p^{s\dagger} e^{ip\cdot x}\right)$$

$$= -i\gamma^2 \psi^*(x) = -i\gamma^2 (\psi^\dagger)^T$$

$$\Rightarrow \quad \boxed{C\psi(x)C = -i(\bar\psi\gamma^0\gamma^2)^T}$$

$$C\bar\psi(x)C = C\psi^\dagger C\gamma^0 = (-i\gamma^2\psi)^T\gamma^0$$

$$\Rightarrow \quad \boxed{C\bar\psi(x)C = (-i\gamma^0\gamma^2\psi)^T}$$

(Note that C is a linear unitary operator even though takes $\psi \to \psi^*$)

Action on bilinears?

SCALAR

$$C\bar{\psi}\psi C = (-i\gamma^0\gamma^2\psi)^T(-i\bar{\psi}\gamma^0\gamma^2)^T$$

$$= -\gamma^0_{ab}\gamma^2_{bc}\psi_c \, \bar{\psi}_d\gamma^0_{de}\gamma^2_{ea}$$

$$= +\bar{\psi}_d\gamma^0_{de}\gamma^2_{ea}\gamma^0_{ab}\gamma^2_{bc}\psi_c$$

$$= \bar{\psi}\gamma^0\gamma^2\gamma^0\gamma^2\psi = -\bar{\psi}\gamma^2\gamma^0\gamma^0\gamma^2\psi$$

$$= \underline{+\bar{\psi}\psi}$$

Similarly one finds:

PSEUDOSCALAR

$$C\, i\bar{\psi}\gamma_5\psi C = \underline{i\bar{\psi}\gamma_5\psi}$$

VECTOR

$$C\bar{\psi}\gamma^\mu\psi C = -\underline{\bar{\psi}\gamma^\mu\psi}$$

AXIAL VECTOR

$$C\bar{\psi}\gamma^\mu\gamma^5\psi C = \underline{+\bar{\psi}\gamma^\mu\gamma^5\psi}$$

Now let's summarize our discrete transformations!

Summary

	$\bar\psi\psi$	$i\bar\psi\gamma_5\psi$	$\bar\psi\gamma^\mu\psi$	$\bar\psi\gamma^\mu\gamma^5\psi$	∂_μ
P	$+1$	-1	$(-1)^\mu$	$-(-1)^\mu$	$(-1)^\mu$
T	$+1$	-1	$(-1)^\mu$	$(-1)^\mu$	$-(-1)^\mu$
C	$+1$	$+1$	-1	$+1$	$+1$
CPT	$+1$	$+1$	-1	-1	-1

Note: $\mathcal{L} = \bar\psi(i\gamma^\mu\partial_\mu - m)$ is CPT invariant

CPT theorem: (Pauli)

One cannot build a Lorentz invariant QFT w/ a Hermitian Hamiltonian that violates CPT.

(See Itzykson + Zuber for sketch of proof!)

Basically want to show that

$$\mathrm{CPT}\,\mathcal{L}(x)\,(\mathrm{CPT})^+ = \mathcal{L}(-x)$$