What about charges carried by fermions and antifermions??

Recall that free Dirac theory has a conserved vector current:

$$\underset{\sim\sim\sim}{j^\mu = \bar{\psi}\gamma^\mu\psi} \qquad (\psi \to e^{i\alpha}\psi)$$

then $$Q = \int d^3x\, \bar{\psi}\gamma^0\psi = \int d^3x\, \psi^\dagger(x)\,\psi(x)$$

Plugging in Fourier expansion of ψ:

$$Q = \int \frac{d^3p}{(2\pi)^3} \sum_s \left(a_p^{s\dagger} a_p^s + b_{-p}^s b_{-p}^{s\dagger} \right)$$

$$\Downarrow$$

$$\boxed{Q = \int \frac{d^3p}{(2\pi)^3} \sum_s \left(a_p^{s\dagger} a_p^s - b_p^{s\dagger} b_p^s \right)} \quad (+\infty)$$

CREATES fermion w/ charge +1

CREATES anti-fermion w/ charge −1

When E+M is included, this is E+M charge!

In Q.E.D., ψ describes electrons + positrons

$$\boxed{\begin{array}{lll} \psi_\alpha(x)\,|0\rangle & \text{creates} & \text{positron at } x \\ \bar{\psi}_\alpha(x)\,|0\rangle & \text{"} & \text{electron at } x \end{array}}$$

The Dirac Propagator

Recall from notes:

$$\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle = (i\not{\partial}_x + m)_{ab}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip\cdot(x-y)}$$

$$\langle 0|\bar{\psi}_a(y)\psi_b(x)|0\rangle = -(i\not{\partial}_x + m)_{ba}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip\cdot(y-x)}$$

As before we can define a Retarded Green's fn:

$$\boxed{\begin{aligned} S_R^{ab}(x-y) &= \theta(x^0-y^0)\langle 0|\{\psi_a(x),\bar{\psi}_b(y)\}|0\rangle \\ &= (i\not{\partial}_x + m)^{ab}\, D_R(x-y) \end{aligned}}$$

$$\left(D_R(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2}e^{-ip\cdot(x-y)} \right)$$

$$\begin{aligned}(i\not{\partial}_x - m)S_R(x-y) &= (-\not{\partial}_x\not{\partial}_x - m^2)\, D_R(x-y) \\ &= -(\Box + m^2)\, D_R(x-y) \\ &= i\,\delta^{(4)}(x-y) \quad \checkmark \end{aligned}$$

S_R is Green's fn. of Dirac operator

Let's find momentum-space representation!

$$S_R(x-y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)}\, \tilde{S}_R(p)$$

$$(i\partial\!\!\!/_x - m)\, S_R(x-y) = \int \frac{d^4p}{(2\pi)^4}\, (p\!\!\!/ - m)\, e^{-ip(x-y)}\, \tilde{S}_R(p)$$

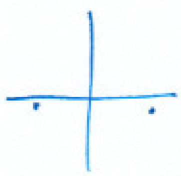
$$= i\,\delta^{(4)}(x-y)$$

$$\Rightarrow \quad \boxed{\tilde{S}_R(p) = \frac{i}{p\!\!\!/ - m} = \frac{i\,(p\!\!\!/ + m)}{p^2 - m^2}}$$

Explicitly:

$$S_R(x-y) = \int \frac{d^4p}{(2\pi)^4}\, \frac{i(p\!\!\!/ + m)}{p^2 - m^2}\, e^{-ip\cdot(x-y)}$$

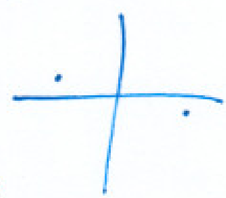
$$\left(= (p_0 + i\epsilon)^2 - p_i p_i - m^2\right)$$



$$\left\{\begin{array}{ll} \text{POLES BELOW!} & \\ x^0 > y^0 & \text{pick up poles} \\ x^0 < y^0 & 0 \end{array}\right\}$$

FEYNMAN

$$\underbrace{S_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\, \frac{i(p\!\!\!/ + m)}{p^2 - m^2 + i\epsilon}\, e^{-ip\cdot(x-y)}}$$



$$= \begin{cases} \langle 0|\, \psi(x)\, \bar{\psi}(y)\, |0\rangle & x^0 > y^0 \quad \text{close below} \\ -\langle 0|\, \bar{\psi}(y)\, \psi(x)\, |0\rangle & x^0 < y^0 \quad \text{close above} \end{cases}$$

$$= \underbrace{\langle 0|\, T\left(\psi(x)\, \bar{\psi}(y)\right) |0\rangle} \qquad (\text{note} - \text{sign})$$

Discrete Symmetries of Dirac Fermions

Lorentz transformations $\Rightarrow$ Unitary operator $U(\Lambda)$

$$U(\Lambda)\,\psi(x)\,U^{-1}(\Lambda) = \Lambda_{\frac{1}{2}}^{-1}\,\psi(\Lambda x)$$

So far we've considered <u>continuous</u> space-time symmetries: can be deformed to the identity operator $\underline{\mathbb{1}}$.

Now let's construct operators that implement <u>discrete symmetries</u>.

2 other spacetime symmetries:

<u>Parity</u>:	P:	$(t, x_i) \to (t, -x_i)$
<u>Time Reversal</u>:	T:	$(t, x_i) \to (-t, x_i)$

There is <u>no continuous</u> Lorentz transformation that implements these symmetries.

However $x_\mu x^\mu = x^2 = t^2 - x_i x_i$ is left invariant by P and T.

Nomenclature (Lorentz Group: L)

$$
\begin{tikzcd}
\text{(orthochronous)}\ L_+^{\uparrow} \arrow[rr, leftrightarrow, "P"] \arrow[d, leftrightarrow, "T"'] & & L_-^{\uparrow} = P L_+^{\uparrow} \arrow[d, leftrightarrow, "T"] \\
\text{(non-orthochronous)}\ L_+^{\downarrow} = T L_+^{\uparrow} \arrow[rr, leftrightarrow, "P"'] & & L_-^{\downarrow} = P T L_+^{\uparrow}
\end{tikzcd}
$$

(proper) (improper)

It is convenient (and deep) to discuss another NON-spacetime symmetry at the same time:

$$\boxed{\text{charge conjugation}: \quad C: \quad \text{particle} \longrightarrow \text{antiparticle}}$$

Sensible relativistic theories must be invariant under $L_+^{\uparrow}$ but need not be invariant under P, T or C.

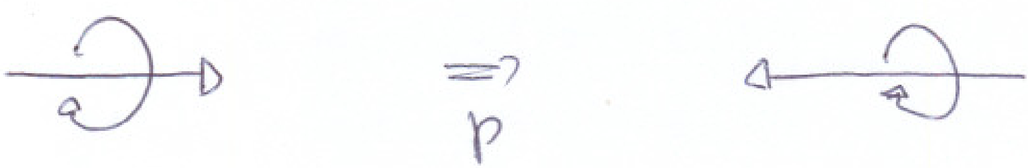
CPT is always conserved! (We will come back to this)

In the <u>Standard Model</u>, the <u>weak interaction</u> breaks P and C.

<u>Rare processes</u> violate CP
(and therefore T)
(seen in Kaon decays)

<u>PARITY</u>

P should reverse momentum but not affect spin.
($J_i = \epsilon_{ijk} x_j p_k$)



Should be implemented by a unitary operator:

$$\underline{U(P) = P}$$

which takes
$$a_{p_i}^{s}|0\rangle \longrightarrow a_{-p_i}^{s}|0\rangle$$

<u>That is</u>:

$$\boxed{P\, a_p^s\, P^{-1} = \eta_a\, a_{-p}^s}$$

PHASE

Note that except:

$$P^2 = \mathbb{1} \Rightarrow P = P^{-1}$$

Similarly,

$$\boxed{P\, b_p^s\, P^{-1} = P\, b_p^s\, P = \eta_b\, b_{-p}^s}$$

As observables are built from even # of ψ's,

$$\eta_a^2 ,\ \eta_b^2 = \pm 1$$

Let's find P in Dirac theory.

P should act in 4×4 space of γ matrices.

$$\boxed{\begin{aligned} P\, \psi(x)\, P = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \Big(& \eta_a\, a_{-\vec{p}}^s\, u^s(p)\, e^{-ip\cdot x} \\ & + \eta_b^*\, b_{-\vec{p}}^{s\dagger}\, v^s(p)\, e^{ip\cdot x} \Big) \end{aligned}}$$

(Heisenberg Rep.)

Change of variables:

$p = (p^0, p^i) \qquad \tilde{p} = (p^0, -p^i)$

Note that $p \cdot x = \hat{p} \cdot (t, -x_i)$

$\{$ Recall: $\sigma^\mu = (1, \sigma^i)$ $\quad \bar{\sigma}^\mu = (1, -\sigma^i) \}$

$$\hat{p} \cdot \sigma = p \cdot \bar{\sigma} \qquad \hat{p} \cdot \bar{\sigma} = p \cdot \sigma$$

$$u(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\, \xi \\ \sqrt{p \cdot \bar{\sigma}}\, \xi \end{pmatrix} = \begin{pmatrix} \sqrt{\hat{p} \cdot \bar{\sigma}}\, \xi \\ \sqrt{\hat{p} \cdot \sigma}\, \xi \end{pmatrix} = \gamma^0 u(\hat{p})$$

Similarly,

$$v(p) = -\gamma^0 v(\hat{p})$$

$$\boxed{P \psi(x) P = \int \frac{d^3 \hat{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\hat{p}}}} \sum_s \left(\eta_a a^s_{\hat{p}} \gamma^0 u^s(\hat{p}) e^{-i\hat{p} \cdot (t, -x_i)} - \eta_b^* b^{s\dagger}_{\hat{p}} \gamma^0 v^s(\hat{p}) e^{i\hat{p} \cdot (t, -x_i)} \right)}$$

WANT: constant matrix $\times$ $\psi(-x_i, t)$

works iff:
$$\boxed{\eta_b^* = -\eta_a}$$

$\{ \Rightarrow \eta_a \eta_b = -\eta_a \eta_a^* = -1 \Rightarrow |\eta_a|^2 = +1 \}$

$$\Rightarrow \quad \boxed{P \psi(t, x_i) P = \eta_a \gamma^0 \psi(t, -x_i)}$$

With this transformation for ψ, how do the bilinears transform??

First,

$$P \bar{\psi}(t, x_i) P = P \psi^{\dagger}(t, x_i) P \gamma^0$$

$$= (P \psi(t, x_i) P)^{\dagger} \gamma^0 \quad \Rightarrow$$

$$\boxed{P \bar{\psi}(t, x_i) P = \eta_a^* \bar{\psi}(t, -x_i) \gamma_0}$$

Thus,

$$P \bar{\psi} \psi P = P \bar{\psi} P P \psi P$$

$$= |\eta_a|^2 \bar{\psi}(t, -x_i) \gamma_0 \gamma_0 \psi(t, -x_i)$$

$$= + \bar{\psi}\psi(t, -x_i)$$

SCALAR

$$P \bar{\psi} \gamma^\mu \psi P = \bar{\psi} \gamma^0 \gamma^\mu \gamma^0 \psi(t, -x_i)$$

$$= \begin{cases} + \bar{\psi} \gamma^\mu \psi(t, -x_i) & \mu = 0 \\ - \bar{\psi} \gamma^\mu \psi(t, -x_i) & \mu = i \end{cases}$$

VECTOR

$$P \, i \bar{\psi} \gamma^5 \psi P = i \bar{\psi} \gamma^0 \gamma^5 \gamma^0 \psi(t, -x_i) = -i \bar{\psi} \gamma^5 \psi(t, -x_i)$$

PSEUDOSCALAR

$$P \bar{\psi} \gamma^\mu \gamma^5 \psi P = \begin{cases} - \bar{\psi} \gamma^\mu \gamma^5 \psi(t, -x_i) & \mu = 0 \\ + \bar{\psi} \gamma^\mu \gamma^5 \psi(t, -x_i) & \mu = i \end{cases}$$

AXIALVECTOR

TIME REVERSAL

Is T a unitary operator ??

WANT: $a_p \to a_{-p}$ ①

$b_p \to b_{-p}$ ②

$\psi(t, x_i) \to \psi(-t, x_i)$

{ i.e. reverse velocities and retrace back to original configuration ... }

But we saw that if ① and ②, then

$\psi(t, x_i) \to \psi(t, -x_i)$

Let's require T to be symmetry of free Dirac Theory:

$$[T, H] = 0 \qquad (*)$$

$$\psi(t, x_i) = e^{iHt} \psi(x_i) e^{-iHt}$$

(4) (5)

From $(*)$,

$$T\psi(t, x_i)T = e^{iHt}(T\psi(x_i)T)e^{-iHt}$$

Now apply to vacuum state w/ $H|0\rangle = 0$

(*) $\underbrace{T\psi(t,x_i)T|0\rangle = e^{iHt}(T\psi(x_i)T)|0\rangle}$

Expect action of T to be such that

$$\psi(-t,x_i)|0\rangle = e^{iHt}\psi(x_i)|0\rangle$$ positive frequency

However we also have (from (4) to (5))

$$\psi(-t,x_i)|0\rangle = e^{-iHt}\psi(x_i)e^{iHt}|0\rangle$$
$$= e^{-iHt}\psi(x_i)|0\rangle$$ negative frequency

$\therefore$ T cannot be implemented as a unitary operator.

In Q.M. T is antiunitary:

$$\langle T\varphi|T\psi\rangle = \langle\psi|\varphi\rangle \quad T^{+}T = \mathbb{1}$$

Hence invariance under T means

$$THT^{+} = H$$

How do we implement this?
Have T act on c-numbers!!

$$\boxed{Tc = c^{*}T}$$ (changes phases)

Now (*) $\Rightarrow$ $\underbrace{T\psi(t,x_i)T|0\rangle = e^{-iHt}(T\psi(x_i)T)|0\rangle}$

change of phase!

$\left\{ \text{This effectively changes sign of } t \text{ in Q.M. as all time evolution is in these phases.} \right\}$

Direction of spin is changed!!

$(J_i = \epsilon_{ijk} x_j p_k)$

Mathematically, T acting on ψ requires object that flips the spinor ξ.

<u>Change of notation</u>: S = physical spin component of fermion along a specific axis.

Consider <u>polar coordinates</u> about this axis:

$$\xi(\uparrow) = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix} \qquad \xi(\downarrow) = \begin{pmatrix} -e^{i\varphi}\sin\theta/2 \\ \cos\theta/2 \end{pmatrix}$$

(Axis w/ polar coordinates θ, φ)

$$\boxed{\xi^s = \left(\xi(\uparrow), \xi(\downarrow)\right) \qquad s = 1,2}$$

Define: $\boxed{\xi^{-s} = -i\sigma^2 (\xi^s)^*}$

$\Rightarrow$ $\boxed{\xi^{-s} = (\xi(\downarrow), -\xi(\uparrow))}$

↑ weird convention

CHECK: $-i\sigma^2(\xi^1)^* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta/2 \\ e^{-i\varphi}\sin\frac{\theta}{2} \end{pmatrix}$

$= \begin{pmatrix} -e^{-i\varphi}\sin\frac{\theta}{2} \\ \cos\theta/2 \end{pmatrix} = \xi^2$ ✓

Two successive flips:

$$(\xi(\uparrow), \xi(\downarrow)) \to (\xi(\downarrow), -\xi(\uparrow))$$

$$\downarrow$$

$$(-\xi(\uparrow), -\xi(\downarrow))$$

$$= (-1)\,(\xi(\uparrow), \xi(\downarrow))$$

Let's associate fermion spin states w/ these spinors: Recall:

a_p^s destroys fermion w/ spinor $u^s(p)$ (ξ^s)

b_p^s " antifermion w/ " $v^s(p)$ (ξ^{-s})

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi^s \\ \sqrt{p\cdot\bar\sigma}\ \xi^s \end{pmatrix} \qquad v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi^{-s} \\ -\sqrt{p\cdot\bar\sigma}\ \xi^{-s} \end{pmatrix}$$

$\left\{ \text{Recall} \quad \begin{array}{l} J_z\, a|0\rangle = \pm\frac{1}{2}\, a|0\rangle \\ J_z\, b|0\rangle = \mp\frac{1}{2}\, b|0\rangle \end{array} \right\}$

In analogy w/ definition of ξ^{-s}, define:

$$a_{p_i}^{-s} = (a_{p_i}^2, -a_{p_i}^1) \qquad b_{p_i}^{-s} = (b_{p_i}^2, -b_{p_i}^1)$$

Now we can work out relation between Dirac spinors u and v and their time reversals.

Define $\tilde p = (p^0, -p^i)$

Using methods applied earlier, can show

$$\boxed{\sqrt{\tilde p\cdot\sigma}\ \sigma^2 = \sigma^2 \sqrt{p\cdot\sigma^*}}$$

$u^{-s}(\tilde p) = u^s(p)$ w/ flipped momentum and spin

$$U^{-s}(\hat{p}) = \begin{pmatrix} \sqrt{\hat{p}\cdot\sigma}\,(-i\sigma^2\xi^{s*}) \\ \sqrt{\hat{p}\cdot\bar{\sigma}}\,(-i\sigma^2\xi^{s*}) \end{pmatrix} = \begin{pmatrix} -i\sigma^2\sqrt{p\cdot\sigma^*}\;\xi^{s*} \\ -i\sigma^2\sqrt{p\cdot\bar{\sigma}^*}\;\xi^{s*} \end{pmatrix}$$

$$= -i\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}[U^s(p)]^*$$

$$= \boxed{-\gamma^1\gamma^3\,[U^s(p)]^* = U^{-s}(\hat{p})}$$

$$\left\{ \underline{\text{Recall}}: \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \qquad \gamma^1\gamma^3 = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix} \right\}$$

Similarly,

$$\boxed{-\gamma^1\gamma^3\,[V^s(p)]^* = V^{-s}(\hat{p})}$$

Now define action of T on Q.M. operators:

$$\boxed{T\,a^s_{p_i}\,T = a^{-s}_{-p_i} \qquad T\,b^s_{p_i}\,T = b^{-s}_{-p_i}}$$

(forget about overall phase)

<u>Now we're ready</u>!!

$$T\,\psi(t,x_i)\,T = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s T\left(a^s_{p_i}u^s(p)\,e^{-ip\cdot x} + b^{s\dagger}_{p_i}v^s(p)\,e^{ip\cdot x}\right)T$$