This is remarkable result!

$$\boxed{\text{Spin is given by representations of the Lorentz Group}}$$

Let's look with more detail at spin of ψ:

Compute the Angular Momentum Operator

Under a Lorentz transformation: (original convention)

$$\boxed{\psi(x) \to \psi'(x) = \Lambda_{\frac{1}{2}} \psi(\Lambda^{-1} x)}$$

Noether's Thm:

Need variation in the field at fixed space-time point:

$$\boxed{\delta\psi = \psi'(x) - \psi(x) = \Lambda_{\frac{1}{2}} \psi(\Lambda^{-1}x) - \psi(x)}$$

Consider infinitesimal rotation by θ about z-axis.

Recall: $$x^\alpha \to \left(\delta^\alpha_\beta - \tfrac{i}{2}\omega_{\mu\nu}\left(\mathcal{J}^{\mu\nu}\right)^\alpha_{\ \beta}\right) x^\beta$$

$$\left(\mathcal{J}^{\mu\nu}\right)^\alpha_{\ \beta} = i\left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha\right)$$

choose: $\omega_{12} = -\omega_{21} = \theta$

Rotation in x-y plane $\Rightarrow$ $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\theta & 0 \\ 0 & \theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$\Lambda_{\frac{1}{2}} = \mathbb{1} - \tfrac{i}{2}\omega_{\mu\nu}S^{\mu\nu} + \mathcal{O}(\omega^2)$$

$$= \mathbb{1} - \tfrac{i}{2}\theta\Sigma^3 \qquad \left(S^{ij} = \tfrac{1}{2}\epsilon^{ijk}\Sigma^k\right)$$

$$\therefore \quad \delta\psi(x) = (\mathbb{1} - \tfrac{i}{2}\theta\Sigma^3)\psi(t, x+\theta y, y-\theta x, z) - \psi(x)$$

$$= (\mathbb{1} - \tfrac{i}{2}\theta\Sigma^3)\left[\psi(x) + \theta y\partial_x\psi(x) - \theta x\partial_y\psi(x)\right] - \psi(x)$$

$$= -\tfrac{i}{2}\theta\Sigma^3\psi(x) + \theta y\partial_x\psi(x) - \theta x\partial_y\psi(x) + \mathcal{O}(\theta^2)$$

$$= -\theta\left[x\partial_y - y\partial_x + \tfrac{i}{2}\Sigma^3\right]\psi(x)$$

$$\delta\psi(x) \equiv \theta\,\Delta\psi$$

time component of Noether current:

$$j^0_z = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)}\Delta_z\psi \qquad \left(\mathcal{L} = \bar\psi i\not\partial\psi + \dots = \bar\psi i\gamma^0\partial^0\psi + \dots\right)$$

$$= -(i\bar\psi\gamma^0)(x\partial_y - y\partial_x + \tfrac{i}{2}\Sigma^3)\psi$$

$$j^0_z = -i\bar\psi\gamma^0(x\partial_y - y\partial_x + \tfrac{i}{2}\Sigma^3)\psi$$

Doing same for rotations about x and y axes:

$$j^0_i = -i\Psi^\dagger[\epsilon_{ijk}x_j\nabla_k + \tfrac{i}{2}\Sigma^3]\Psi$$

$\Rightarrow$ Angular momentum operator

Spin A.M.

$$\boxed{J_i = \int d^3x\, j^0_i = \int d^3x\, \Psi^\dagger[\epsilon_{ijk}x_j(-i\nabla_k) + \tfrac{1}{2}\Sigma_i]\Psi}$$

Orbital A.M. in N.R. Limit.

$$\left\{\begin{array}{l}\text{One can think of this as arising from} \\ J^{\mu\nu} = \int d^3x\,(x^\mu T^{0\nu} - x^\nu T^{0\mu}) \\ \text{w/}\quad J_k = \tfrac{1}{2}\epsilon_{ijk}J^{ij}\end{array}\right\}$$

For relativistic fermions decomposition into "Orbital" and "Spin" parts is not straightforward.

Now let's show that Dirac particle has spin $\frac{1}{2}$.

Consider particle <u>at rest</u> so that 1st term doesn't contribute.

$$J_3 = \int d^3x\ \psi^\dagger(x_i)\ \frac{1}{2}\Sigma_i\ \psi(x_i)$$

(schrödinger picture)

Expanding in "ladder" operators a, b:

$$J_3 = \int d^3x \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{\sqrt{2E_p 2E_{p'}}} e^{-ip_i' x_i} e^{ip_i x_i} \sum_{r,r'}$$

$$\times \left[a_{p'}^{r'\dagger} u^{r'\dagger}(p_i') + b_{p'}^{r'} v^{r'}(-p_i') \right] \frac{\Sigma^3}{2} \left[a_p^r u^r(p_i) + b_{-p}^{r\dagger} v^r(-p_i) \right]$$

We want to apply this operator to the 1-particle, zero-momentum state $a_0^{s\dagger}|0\rangle$.

$$J_3\left(a_0^{s\dagger}|0\rangle\right) = [J_3, a_0^{s\dagger}]|0\rangle$$

Only $[a_p^{r\dagger} a_{p'}^{r'}, a_0^{s\dagger}]$ contributes:

$$\Rightarrow J_3\, a_0^{s\dagger}|0\rangle = \frac{1}{2m}\sum_r u^{r\dagger}(0) \frac{\Sigma^3}{2} u^s(0)\, a_0^{r\dagger}|0\rangle$$

$$\left\{ \begin{array}{l} \text{In rest frame: } u^r(p_0) = \sqrt{m}\begin{pmatrix} \xi^r \\ \xi^r \end{pmatrix} \\ \Sigma^3 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} \end{array} \right\}$$

$$\Rightarrow \quad J_3 a_0^{s+}|0\rangle = \sum_r \left(\xi^{r+} \frac{\sigma^3}{2} \xi^s\right) a_0^{r+}|0\rangle$$

Choose ξ^s to be eigenstate of σ^3:

$$\boxed{\begin{aligned} J_3 \left(a_0^{s+}|0\rangle\right) &= +\tfrac{1}{2}\left(a_0^{s+}|0\rangle\right) \quad \text{for } \xi^s = \begin{pmatrix}1\\0\end{pmatrix} \\ J_3 \left(a_0^{s+}|0\rangle\right) &= -\tfrac{1}{2}\left(a_0^{s+}|0\rangle\right) \quad \text{for } \xi^s = \begin{pmatrix}0\\1\end{pmatrix} \end{aligned}}$$

These are "electrons"

Similarly can show:

$$\boxed{\begin{aligned} J_3 \left(b_0^{s+}|0\rangle\right) &= -\tfrac{1}{2}\left(b_0^{s+}|0\rangle\right) \quad \text{for } \xi^s = \begin{pmatrix}1\\0\end{pmatrix} \\ J_3 \left(b_0^{s+}|0\rangle\right) &= +\tfrac{1}{2}\left(b_0^{s+}|0\rangle\right) \quad \text{for } \xi^s = \begin{pmatrix}0\\1\end{pmatrix} \end{aligned}}$$

These are "positrons"