Why does this make sense??

Consider general Q.M. algebra:

$$\{b, b^+\} = 1 \qquad \{b, b\} = \{b^+, b^+\} = 0$$

Say $b|0\rangle = 0 \qquad b^+|0\rangle = |1\rangle$

Then $b|1\rangle = |0\rangle \qquad b^+|1\rangle = b^+b^+|0\rangle = 0$

Hilbert space $\Rightarrow$ $\boxed{\text{2 states : } |0\rangle, |1\rangle}$

Interpretation:

(A) $|0\rangle$ is empty and b^+ fills it

OR

(B) $|1\rangle$ is empty $b \; (\equiv \tilde{b}^+)$ fills it

Descriptions are equivalent

Spin and Statistics

Consider state with no spin ½ particle : $|0\rangle$

$a_\alpha^\dagger$; operator that creates spin ½ particle with quantum numbers α.

$a_\alpha^\dagger |0\rangle$: state with spin ½ particle with quantum #'s α.

Suppose want to have another spin ½ particle with quantum #'s β:

$$a_\beta^\dagger a_\alpha^\dagger |0\rangle$$

For this state to be antisymmetric under interchange of α and β must have

$$\boxed{\{a_\beta^\dagger, a_\alpha^\dagger\} = 0}$$

Hermitian conjugation ⇒

$$\boxed{\{a_\beta, a_\alpha\} = 0}$$

Now say the number operator is given by

$$\boxed{N = a_\alpha^+ a_\alpha}$$ (w/ summation convention)

Expect: $N|0\rangle = 0$, $N\,(a_\beta^+|0\rangle) = (a_\beta^+|0\rangle)$

$$L.H.S = [N, a_\beta^+]|0\rangle = [a_\alpha^+ a_\alpha, a_\beta^+]|0\rangle$$

$$= \left[a_\alpha^+\{a_\alpha, a_\beta^+\} - \underbrace{\{a_\alpha^+, a_\beta^+\}}_{0} a_\alpha\right]|0\rangle$$

$$= a_\alpha^+\{a_\alpha, a_\beta^+\}|0\rangle$$

$$= R.H.S \quad \text{If}$$

$$\boxed{\{a_\alpha, a_\beta^+\} = \delta_{\alpha\beta}}$$

(Generalizes to continuous labels; i.e. momentum)

Holds for multiparticle field operator:

$$\boxed{a_p^{+} a_q^{+} |0\rangle = -a_q^{+} a_p^{+} |0\rangle}$$

If creation/annihilation operators obey anticommutation relations, particles obey Fermi-Dirac statistics

General Theorem (due to Pauli)

Lorentz Invariance + positive energies + positive norms + causality

$\Downarrow$

particles w/ integer spin $\Rightarrow$ Bose-Einstein statistics

particles w/ half-integer spin $\Rightarrow$ Fermi-Dirac statistics

Let's sum things up: CORRECT QUANTIZATION

(Drop tilde on b's)

Heisenberg picture operators:

$$\boxed{\begin{aligned}\psi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(a_p^s u^s(p) e^{-ip\cdot x} + b_p^{s\dagger} v^s(p) e^{ip\cdot x} \right)\\ \bar{\psi}(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(b_p^s \bar{v}^s(p) e^{-ip\cdot x} + a_p^{s\dagger} \bar{u}^s(p) e^{ip\cdot x} \right)\end{aligned}}$$

$$\boxed{\begin{gathered}\{ a_p^r, a_q^{s\dagger} \} = \{ b_p^r, b_q^{s\dagger} \} = (2\pi)^3 \delta^{(3)}(p_i - q_i) \delta^{rs}\\ \text{else} = 0\end{gathered}}$$

$$\boxed{\{ \psi_a(x_i), \psi_b(y_i)^\dagger \} = \delta^{(3)}(x_i - y_i) \delta_{ab}}$$

$$a_p^s |0\rangle = b_p^s |0\rangle = 0$$

$H = ??$

$$\boxed{H = \int \frac{d^3p}{(2\pi)^3} \sum_s \underbrace{E_p} \left(a_p^{s\dagger} a_p^s + b_p^{s\dagger} b_p^s \right) \quad (+\infty)}$$

Vacuum is state of lowest energy:

$$H|0\rangle = 0$$

Momentum operator ??

$$P^i = \int d^3x \, \psi^\dagger (-i\nabla_i) \psi \qquad \left(= \int T^{0i} d^3x \right)$$

$$\Rightarrow \boxed{P^i = \int \frac{d^3p}{(2\pi)^3} \sum_s \underbrace{p_i} \left(a_p^{s\dagger} a_p^s + b_p^{s\dagger} b_p^s \right)}$$

$a_p^{s\dagger}$ and $b_p^{s\dagger}$ create particles w/ energy $+\underbrace{E_p}$ and momentum $\underbrace{p_i}$.

$a_p^{s\dagger}$:	CREATES	FERMIONS
$b_p^{s\dagger}$:	CREATES	ANTI FERMIONS

One-particle States

$$|p_i, s\rangle = \sqrt{2E_p}\; a_p^{s\dagger} |0\rangle$$

$$\langle p_i, r | q_i, s\rangle = \sqrt{2E_p}\sqrt{2E_q}\,\langle 0| a_p^r a_q^{s\dagger} |0\rangle$$

$$= \sqrt{4E_pE_q}\,\langle 0| \{a_p^r, a_q^{s\dagger}\} |0\rangle$$

$$= (2\pi)^3 \sqrt{4E_pE_q}\,\delta^{(3)}(p_i - q_i)\,\delta^{rs}$$

$$\Rightarrow \boxed{\langle p_i, r | q_i, s\rangle = 2E_p (2\pi)^3 \delta^3(p_i - q_i)\,\delta^{rs}}$$

Lorentz invariant normalization

$\Downarrow$

This means that the operator $U(\Lambda)$ that implements Lorentz transformation on quantum states is unitary (even though $\Lambda_{\frac{1}{2}}$ is not).

$$\left\{\begin{aligned} U(\Lambda)|q,s\rangle &= |\Lambda q, s\rangle = |q', s\rangle \\ \langle p', r | q', s\rangle &= \langle p, r| U^\dagger(\Lambda)\, U(\Lambda) | q, r\rangle \\ &= \langle p, r | q, r\rangle \checkmark \end{aligned}\right\}$$

$$\therefore \quad \boxed{U^\dagger(\Lambda)\, U(\Lambda) = \mathbb{1}}$$

(This is what we expect in the quantum theory)

Can show that:

$$\boxed{U(\Lambda)\, \psi(x)\, U^{-1}(\Lambda) = \Lambda_{\frac{1}{2}}^{-1}\, \psi(\Lambda x)}$$

Recall that classically we had:

$$\psi(x) \longrightarrow \Lambda_{\frac{1}{2}}\, \psi(\Lambda^{-1} x)$$

transformation seems to be in wrong direction

BEFORE: transformed pre-existing field distribution

NOW: transform action of ψ in creating or destroying particles

2 ways of implementing Lorentz transformations (e.g. active vs. passive)

Only relative orientation is physically relevant!

Poincaré Group (part II)

(In context of Quantum Mechanics)

What is spin ??

Does it have something to do with angular momentum ?

what is the angular momentum operator ?

Recall the Poincaré algebra:

$$[J_{\mu\nu}, J_{\rho\sigma}] = i(g_{\nu\rho} J_{\mu\sigma} - g_{\mu\rho} J_{\nu\sigma} + g_{\mu\sigma} J_{\nu\rho} - g_{\nu\sigma} J_{\mu\rho})$$

$$[P_\mu, J_{\rho\sigma}] = i(g_{\mu\rho} P_\sigma - g_{\mu\sigma} P_\rho)$$

$$[P_\mu, P_\nu] = 0$$

$J_i, K_i \in J_{\mu\nu}$ (Rotations + Boosts)

$H, P_i \in P_\mu$ (space-time translations)

Basic Poincaré transformation:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \qquad (\Lambda^\mu{}_\rho \Lambda^\sigma{}_\mu = \delta^\sigma_\rho)$$

Quantum Lorentz transformations and the Wigner method

The components of P^μ all commute with one another $\Rightarrow$ natural to express physical state vectors in terms of eigenvectors of four-momentum:

$$P^\mu |p\rangle = p^\mu |p\rangle \qquad (*)$$

Say $\quad U(\Lambda, a)\,|p\rangle = |\Lambda p\rangle$

Then,

$$P^\mu |\Lambda p\rangle = (\Lambda p)^\mu |\Lambda p\rangle$$

Now $\quad (\Lambda p)^2 = (\Lambda p)^\mu (\Lambda p)_\mu$

$$= (\Lambda^\mu{}_\rho\, p^\rho)(\Lambda^\sigma{}_\mu\, p_\sigma) = \delta^\sigma_\rho\, p^\rho p_\sigma = p^2$$

$$= P^2 \qquad (\text{by } (*))$$

$\therefore$ Lorentz transformations leave $P^2 = P^\mu P_\mu$ invariant.

(Follows from fact that P^2 commutes with all generators of the Poincaré group)

p^2 is a Casimir operator

$$\boxed{C_1 = p^\mu p_\mu}$$

(commutes w/ all other generators)

Hence, all states obtained by Lorentz transformation from some initial state have same value of p^2.

As sign of p_0 is unchanged by Lorentz transformation we can give complete list of states forming bases for representations of the group:

6 distinct classes $\{p_\mu\}$

(i)	$p^2 = m^2 > 0$,	$p^0 > 0$	physical massive particles
(ii)	$p^2 = m^2 > 0$,	$p^0 < 0$	?
(iii)	$p^2 = 0$	$p^0 > 0$	physical massless particles
(iv)	$p^2 = 0$	$p^0 < 0$	?
(v)	$p^\mu = 0$		the vacuum
(vi)	$p^2 < 0$		virtual particles w/ space-like momenta

Once p^μ in class $\{p^\mu\}$ is chosen there is <u>subgroup</u> of Poincaré group that leaves p^μ invariant:

$$\boxed{\text{Little group of } p^\mu}$$

(similar structure for all momenta in $\{p^\mu\}$)

Consider class (i) : $p^2 = m^2$

In <u>rest frame</u> p^μ is:

$$k^\mu = (m, 0, 0, 0)$$ ("standard" 4-momentum)

What is its <u>little group</u>?

<u>The rotation group $SU(2)$</u> !!

<u>CLAIM</u>: For a time-like momentum, to know effect of arbitrary Lorentz transformation requires <u>only</u> knowledge of representations of the rotation group (little group)

= <u>Main conclusion of Wigner's work !!</u>

Let's demonstrate this!

Consider arbitrary timelike p^μ $(p^2 > 0)$.

Clearly there is Lorentz transformation that takes k^μ to p^μ:

$$\boxed{p^\mu = L^\mu{}_\nu(p)\, k^\nu}$$

{ Generally $L \sim R^{-1} B R$ }

B: standard boost

R: $\hat{p}$ to $\hat{z}$

States in the Hilbert space:

$|p, \sigma)$, $|k, \sigma)$

σ: spin (not necessarily $\frac{1}{2}$!)

$$\boxed{|p, \sigma) = U(L(p))\, |k, \sigma)}$$

$$(U^\dagger U = \mathbb{1})$$

Consider arbitrary Lorentz transformation Λ:

$$p^\mu \to \Lambda^\mu{}_\nu p^\nu$$

$$|p,\sigma\rangle \to \underbrace{U(\Lambda)\,|p,\sigma\rangle}_{\text{Let's find this!}}$$

$$U(\Lambda)\,|p,\sigma\rangle = U(\Lambda)\,U(L(p))\,|k,\sigma\rangle$$

$$= \left[U(L(\Lambda p))\,U^{-1}(L(\Lambda p))\right] U(\Lambda)\,U(L(p))\,|k,\sigma\rangle$$

USE GROUP LAWS:

① $U^{-1}(A) = U(A^{-1})$

② $U(A)\,U(B)\,U(C) = U(ABC)$

$$= U(L(\Lambda p))\,U(L^{-1}(\Lambda p))\,U(\Lambda)\,U(L(p))\,|k,\sigma\rangle$$

(used ①)

$$= U(L(\Lambda p))\,U(L^{-1}(\Lambda p)\,\Lambda\,L(p))\,|k,\sigma\rangle$$

(used ②)

But $L^{-1}(\Lambda p)\,\Lambda\,L(p)$ takes $k^\mu \to k^\mu$!!

$k \to p$

$p \to \Lambda p$

$\Lambda p \to k$

$\therefore$ $L^{-1}(\Lambda p)\,\Lambda\, L(p)$ is a rotation!!

That is,

$$\underbrace{U(L^{-1}(\Lambda p)\,\Lambda\, L(p)) \sim e^{iJ_i\theta_i}}$$

We can write

$$U(\Lambda)\,|p,\sigma\rangle = U(L(\Lambda p)) \underbrace{\sum_{\sigma'} D_{\sigma'\sigma}(R)}\,|k,\sigma'\rangle$$

$\left\{ \begin{array}{c} \text{matrix elements of } e^{iJ_i\theta_i} \text{ w/} \\ R = L^{-1}(\Lambda p)\,\Lambda\, L(p) \end{array} \right\}$

$$= \sum_{\sigma'} D_{\sigma'\sigma}(R)\,\big(U(L(\Lambda p))\,|k,\sigma'\rangle \big)$$

$\{$ Recall $|p,\sigma\rangle = U(L(p))\,|k,\sigma\rangle$ $\}$

$$\Rightarrow \boxed{U(\Lambda)\,|p,\sigma\rangle = \sum_{\sigma'} D_{\sigma'\sigma}(R)\,|\Lambda p,\sigma'\rangle}$$

$\therefore$ To know representations of Lorentz group for timelike states, need only know representations of the rotation group, $SU(2) \sim SO(3)$, the little group.