Therefore, even at level of $H$ we see that quantized Dirac theory is SICK.

Let's go on anyway and look at $\boxed{\text{causality.}}$

Go to Heisenberg picture

Recall:
$$\left\{\begin{aligned} e^{iHt} a_p^s e^{-iHt} &= a_p^s e^{-iE_p t} \\ e^{iHt} b_p^s e^{-iHt} &= b_p^s e^{+iE_p t} \end{aligned}\right\}$$

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left( a_p^s u^s(p) e^{-ip\cdot x} + b_p^{s\dagger} v^s(p) e^{ip\cdot x} \right)$$

$$\bar\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left( a_p^{s\dagger} \bar u^s(p) e^{ip\cdot x} + b_p^s \bar v^s(p) e^{-ip\cdot x} \right)$$

Check that commutator vanishes outside the light cone.

$$[\psi_a(x), \bar\psi_b(y)] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p}$$

$$\times \left[ (\not p + m)_{ab} e^{-ip\cdot(x-y)} + (\not p - m)_{ab} e^{ip\cdot(x-y)} \right]$$

$\Rightarrow$

$$[\psi_a(x), \bar{\psi}_b(y)] = (i\not\partial_x + m)_{ab}\,[\varphi(x), \varphi(y)]$$

As vanishes outside light-cone,
so does $[\psi, \bar{\psi}]$ !!

Let's investigate further:

Say $|0\rangle$ is state such that

$$a_p^s|0\rangle = b_p^s|0\rangle = 0$$

$$[\psi_a(x), \bar{\psi}_b(y)] = \langle 0|\,[\psi_a(x), \bar{\psi}_b(y)]\,|0\rangle$$

↑ c-number

$$= \underbrace{\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle}_{\sim aa^\dagger,\ bb^\dagger \ \ne 0} - \underbrace{\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle}_{\sim a^\dagger a,\ b^\dagger b \ = 0}$$

In K-G case these 2 pieces cancel!

$$\boxed{\begin{array}{ll} D(x-y) & \text{particle from } y \text{ to } x \\ - & \\ D(y-x) & \text{antiparticle from } x \text{ to } y \quad = 0 \end{array}}$$

with $\psi$, cancellation occurs within

$\langle 0 | \psi(x) \bar{\psi}(y) | 0 \rangle$.

$+E_p$ particle from $y$ to $x$

$-E_p$ $\overline{\text{particle}}$ from $y$ to $x$

$= 0$

Can we find a way for $+E_p$ particles to propagate in both directions??

(like K-G field)

THROW AWAY: commutators of $\psi, \bar{\psi}, a, b$
and $a|0\rangle = b|0\rangle = 0$

KEEP: Fourier expansion of $\psi(x)$.

Consider $\langle 0 | \psi(x) \bar{\psi}(y) | 0 \rangle$

(which before contained $+E$ and $-E$ particles)

We would like this matrix element to account for $+E$ particles ONLY.

That is, want $\bar{\psi}(y)|0\rangle \Rightarrow +E$ only

$\Rightarrow$ only $a_p^{s\dagger}$ should contribute.

$$\Rightarrow \quad \underline{\underline{b_p^{s\dagger}|0\rangle = 0}}$$

$$\left\{ \text{And,} \quad \underline{\underline{\langle 0| b_p^{s\dagger} = 0}} \right\}$$

Hence,

$$(*) \quad \boxed{\begin{aligned} \langle 0|\psi(x)\bar{\psi}(y)|0\rangle = \langle 0| &\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_r a_p^r u^r(p)\, e^{-ip\cdot x} \\ &\times \int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}}\sum_s a_q^{\dagger s}\,\bar{u}^s(q)\, e^{iq\cdot x}|0\rangle \end{aligned}}$$

$$\sim \langle 0| a_p^r a_q^{s\dagger}|0\rangle$$

What do we know about this matrix element?

Ground state should be invariant under space translations:

$$|0\rangle = e^{iP_i x_i}|0\rangle$$

Recall: $P^\mu = (H, P^i)$ Poincaré generator

$P_i = \int \frac{d^3p}{(2\pi)^3}\, p_i\, a_p a_p^\dagger$ for K-G field

$$\langle 0| a_p^r a_q^{s\dagger}|0\rangle = \langle 0| a_p^r a_q^{s\dagger} e^{iP_i x_i}|0\rangle$$

$$= e^{i(p_i - q_i)x_i}\langle 0|(a_p^r e^{-ip_i x_i})(a_q^{s\dagger} e^{iq_i x_i}) e^{iP_i x_i}|0\rangle$$

$$= e^{i(p_i - q_i)x_i}\langle 0|(e^{iP_i x_i} a_p^r e^{-iP_i x_i})(e^{iP_i x_i} a_q^{s\dagger} e^{-iP_i x_i}) e^{iP_i x_i}|0\rangle$$

$$= e^{i(p_i - q_i)x_i}\langle 0| e^{iP_i x_i} a_p^r a_q^{s\dagger}|0\rangle$$

$$\Rightarrow \boxed{\langle 0| a_p^r a_q^{s\dagger}|0\rangle = e^{i(p_i - q_i)x_i}\langle 0| a_p^r a_q^{s\dagger}|0\rangle}$$

Under $d^3x$ integral $\Rightarrow$ If $\langle 0| a_p^r a_q^{s\dagger}|0\rangle \neq 0$,

$$p_i = q_i$$

Rotational Invariance ⇒

$$\langle 0 | a_p^r a_q^{s\dagger} | 0 \rangle \neq 0 \Rightarrow \underline{\underline{r = s}}$$

∴ EXPECT

$$\boxed{\langle 0 | a_p^r a_q^{s\dagger} | 0 \rangle = (2\pi)^3 \delta^{(3)}(p_i - q_i)\, \delta^{rs}\, A(p_i)}$$

Normalization

Arbitrary fn. $> 0$

Back to ⊛ :

$$\langle 0 | \psi(x) \bar{\psi}(y) | 0 \rangle = \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{\sqrt{2E_p}\sqrt{2E_q}}$$

$$\sum_{r,s} \langle 0 | a_p^r a_q^{s\dagger} | 0 \rangle\, u^r(p)\, \bar{u}^s(q)\, e^{-ip\cdot x}\, e^{iq\cdot y}$$

$$= \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (\not{p} + m)\, A(p_i)\, e^{-ip\cdot(x-y)}}$$

Lorentz invariant only if $\underline{A(p_i) = A(p^2)}$ is SCALAR

Finally we have:

$$\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle = (i\not{\partial}_x + m)_{ab}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\, e^{-ip(x-y)}\; \underline{\underline{A}}$$

what about $\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle$ ??

Now we want

$$\underline{\underline{b_p^s|0\rangle \neq 0}}\,, \quad a_p^s|0\rangle = 0$$

$$\langle 0|\,\underline{\underline{b_p^{s+} \neq 0}}$$

By similar arguments:

$$\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle = -(i\not{\partial}_x + m)_{ab}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\, e^{+ip\cdot(x-y)}\; \underline{\underline{B}}$$

important !!
(from $\Sigma\, v\bar{v}$)

Schematically,

$$[\psi, \bar{\psi}] = A\, D(x-y) + B\, D(y-x)$$

These pieces would cancel if $\underline{\underline{A = -B}}$

But this is impossible as $A, B > 0$ !!

what really want is:

Outside the light cone:

$$\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle = -\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle$$

$$D(y-x) \qquad\qquad -(-D(y-x))$$

$$\Rightarrow \boxed{\langle 0|\{\psi_a(x),\bar{\psi}_b(y)\}|0\rangle = 0}$$

OUTSIDE LIGHT CONE

Spinors should anticommute at spacelike separation.

Sufficient to preserve causality since all OBSERVABLES

ARE build out of even #'s of spinors.

What about the negative energy problem??

POSTULATE: Anticommutation relations!

$$\boxed{\begin{aligned}\{\psi_a(x_i),\psi_b^\dagger(y_i)\} &= \delta^{(3)}(x_i-y_i)\,\delta_{ab}\\ \{\psi_a(x_i),\psi_b(y_i)\} &= \{\psi_a^\dagger(x_i),\psi_b^\dagger(y_i)\} = 0\end{aligned}}$$

$$\psi(x_i) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}e^{ip_i x_i}\sum_{s=1,2}\left[a_p^s u^s(p_i) + b_{-p}^s v^s(p_i)\right]$$

Consistency then requires:

$$\boxed{\begin{gathered}\{a_p^r, a_p^{s\dagger}\} = \{b_p^r, b_q^{s\dagger}\} = (2\pi)^3 \delta^{(3)}(p_i - q_i)\, \delta^{rs} \\ (\text{else} = 0)\end{gathered}}$$

Again one finds:

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s \left( E_p\, a_p^{s\dagger} a_p^s - E_p\, b_p^{s\dagger} b_p^s \right)$$

$b_p^{s\dagger}$ still creates negative energy!

Let's look more closely.

Define $\hat{b}_p^r \equiv b_p^{s\dagger} \qquad \hat{b}_p^{s\dagger} = b_p^s$

Then $b_p^r b_q^{s\dagger} = -b_q^{s\dagger} b_p^r + \text{junk}$

$$\Rightarrow \quad -E_p\, b_p^{s\dagger} b_p^s = E_p\, b_p^s b_p^{s\dagger} + \text{junk}$$

$$-E_p\, b_p^{s\dagger} b_p^s = E_p\, \hat{b}_p^{s\dagger} \hat{b}_p^s + \text{junk}$$

Choose $|0\rangle$ such that $a_p^s|0\rangle = \hat{b}_p^s|0\rangle = 0$

Then all excitations on $|0\rangle$ have **positive** energy.