u's and v's are orthogonal

$$\boxed{\bar{u}^r(p)\, v^s(p) = \bar{v}^r(p)\, u^s(p) = 0}$$

Note:

$$u^{r\dagger}(p)\, v^s(p) = \left(\xi^{r\dagger}\sqrt{p\cdot\sigma},\ \xi^{r\dagger}\sqrt{p\cdot\bar{\sigma}} \right) \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta^s \end{pmatrix}$$

$$= \xi^{r\dagger}\eta^s \left(p\cdot\sigma - p\cdot\bar{\sigma} \right)$$

$$= 2\,\xi^{r\dagger}\eta^s\, \sigma_i p_i \neq 0$$

But $u^{r\dagger}(p_i)\, v^s(-p_i) = v^{r\dagger}(-p_i)\, u^s(p_i) = 0$

---

what does $\pm$ frequency mean??

Recall:

$$\mathcal{L} = \bar{\psi} i\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi$$

$$\Downarrow$$

$$(i\gamma^\mu\partial_\mu - m)\psi = 0$$

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\psi^\dagger \qquad \mathcal{H} = \pi\dot{\psi} - \mathcal{L}$$

Schematically,

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L}$$

$$= i\bar{\psi}\gamma_0\partial_0\psi - \bar{\psi}i\gamma^\mu\partial_\mu\psi + m\bar{\psi}\psi$$

$$= i\bar{\psi}\gamma^i\partial_i\psi + m\bar{\psi}\psi$$

$$\Downarrow \text{ D.E.}$$

$$= i\bar{\psi}\gamma^0\partial_0\psi$$

$$\mathcal{H} = i\psi^\dagger\frac{\partial}{\partial t}\psi$$

It is clear that $-$ frequency solution

$$\psi(x) = V(p)\,e^{+ip\cdot x} = V(p)\,e^{ip_0t - \vec{p}\cdot\vec{x}}$$

$\Rightarrow$ negative contribution to $\mathcal{H}$!! (not positive definite)

We will have to quantize Dirac field in order to deal w/ this problem.

When we do Feynman diagrams it is useful to sum over polarization states:

E.g. $$\sum_{s=1,2} u^s(p)\,\bar{u}^s(p) = \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s \end{pmatrix} \left(\xi^{s\dagger}\sqrt{p\cdot\bar\sigma},\ \xi^{s\dagger}\sqrt{p\cdot\sigma}\right)$$

$$= \sum_s \xi^s \xi^{s\dagger} \begin{pmatrix} \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\sigma} \end{pmatrix}$$

$$= \underline{1} \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar\sigma & m \end{pmatrix}$$

$$= m\,\underline{1} + \gamma\cdot p$$

2) $$\boxed{\begin{aligned} \sum_s u^s(p)\,\bar{u}^s(p) &= \gamma\cdot p + m \equiv \not{p} + m \\ \sum_s v^s(p)\,\bar{v}^s(p) &= \gamma\cdot p - m = \not{p} - m \end{aligned}}$$

Add to your toolbox!

$$\gamma\cdot A \equiv \gamma_\mu A^\mu \equiv \not{A}$$

Dirac Bilinears

Previously we showed that $\bar{\psi}\psi$ is a Lorentz scalar

$$\left\{ \psi \to \Lambda_{\frac{1}{2}} \psi \qquad \bar{\psi} \to \bar{\psi} \Lambda_{\frac{1}{2}}^{-1} \right\}$$

We also used the fact that $\bar{\psi}\gamma_\mu\psi$ is a 4-vector, in constructing $\mathcal{L}$.

Consider $\boxed{\bar{\psi}\,\Gamma\,\psi}$

Any constant 4×4 matrix

Let's decompose this object into operators with definite Lorentz transformation properties.

$\Gamma =$

	#
$\mathbb{1}$	1
γ^μ	4
$\gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu] = \gamma^{[\mu}\gamma^{\nu]} = -i\sigma^{\mu\nu}$	6
$\circledast \left\{ \gamma^{\mu\nu\rho} = \gamma^{[\mu}\gamma^\nu\gamma^{\rho]} \right.$	4
$\quad\;\; \gamma^{\mu\nu\rho\sigma} = \gamma^{[\mu}\gamma^\nu\gamma^\rho\gamma^{\sigma]}$	1
(anti-symmetric tensors)	16

Easy to check that these tensors transform as expected under Lorentz transformations:

E.g. $\bar{\psi}\gamma^{\mu\nu}\psi \to \frac{1}{2}(\bar{\psi}\Lambda_{\frac{1}{2}}^{-1})(\gamma^\mu \Lambda_{\frac{1}{2}}\Lambda_{\frac{1}{2}}^{-1}\gamma^\nu - \gamma^\nu \Lambda_{\frac{1}{2}}\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu)(\Lambda_{\frac{1}{2}}\psi)$

$= \Lambda^\mu{}_\alpha \Lambda_\beta{}^\nu \bar{\psi}\gamma^{\alpha\beta}\psi$

(where used $\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu\gamma^\nu$)

Can simplify $\gamma^{\mu\nu\rho}$, $\gamma^{\mu\nu\rho\sigma}$ by introducing

$$\boxed{\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma}$$

Say $\gamma^{\mu\nu\rho\sigma} = \epsilon^{\mu\nu\rho\sigma} A$

$4!\,A = \epsilon_{\mu\nu\rho\sigma}\gamma^{\mu\nu\rho\sigma}$

$= 4!\,\gamma^0\gamma^1\gamma^2\gamma^3$

$\Rightarrow A = \gamma^0\gamma^1\gamma^2\gamma^3$

$= -i\gamma^5$

$\therefore \boxed{\gamma^{\mu\nu\rho\sigma} = -i\,\epsilon^{\mu\nu\rho\sigma}\gamma^5}$

Similarly,

$$\boxed{\gamma^{\mu\nu\rho} = -i\,\epsilon^{\mu\nu\rho\sigma}\,\gamma_\sigma\gamma^5}$$

γ_5 Properties

Use $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ $\qquad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

$$\gamma^{0\dagger} = \gamma^0 \quad , \quad \gamma^{i\dagger} = -\gamma^i$$

So $\gamma_5^\dagger = -i(-\gamma^3)(-\gamma^2)(-\gamma^1)\gamma^0$

$= i\gamma^3\gamma^2\gamma^1\gamma^0 = -i\gamma^3\gamma^2\gamma^0\gamma^1$

$= i\gamma^0\gamma^1\gamma^2\gamma^3 \Rightarrow$

$$\boxed{\gamma^{5\dagger} = \gamma^5}$$

Also can easily show

$$\boxed{(\gamma^5)^2 = \mathbb{1}} \qquad \boxed{\{\gamma^5, \gamma^\mu\} = 0}$$

γ^5 notation appropriate as $\gamma^0, \gamma^1, \gamma^2, \gamma^3, \gamma^5$ satisfy Clifford algebra, $\{\gamma, \gamma\} = 2g$ in 5-d

Explicit representation of γ_5??

$$\gamma^5 = i\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}$$

$$= i\begin{pmatrix} -\sigma^1 & 0 \\ 0 & \sigma_1 \end{pmatrix}\begin{pmatrix} -\sigma^2\sigma^3 & 0 \\ 0 & -\sigma^2\sigma^3 \end{pmatrix}$$

$$= i\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

Recap Table of Bilinears

$\Gamma =$

$\mathbb{1}$	SCALAR	1
γ^μ	VECTOR	4
$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$	TENSOR	6
$\gamma^\mu\gamma^5$	AXIAL VECTOR	4
γ^5	PSEUDO SCALAR	1
		16

γ^5 under Lorentz transformations?

$$\gamma^5 = -i/4! \; \epsilon^{\mu\nu\rho\sigma} \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma$$

$$\gamma^5 \longrightarrow \frac{-i}{4!} \epsilon^{\mu\nu\rho\sigma} \Lambda_\mu{}^\kappa \Lambda_\nu{}^\delta \Lambda_\rho{}^\epsilon \Lambda_\sigma{}^\varphi \; \gamma_\kappa \gamma_\delta \gamma_\epsilon \gamma_\varphi$$

$$\Rightarrow \quad \boxed{\gamma^5 \longrightarrow \det(\Lambda)\,\gamma^5}$$

E.g. $\bar{\psi}\gamma^\mu\gamma^5\psi \rightarrow \det(\Lambda)\,\Lambda^\mu{}_\nu\,\bar{\psi}\gamma^\nu\gamma^5\psi$

{ We have implicitly chosen phase $\det(\Lambda) = +1$
Continuous transformations $\Rightarrow$ $\Lambda^0{}_0$ & $\det(\Lambda)$
must have same sign. }

What currents can we form out of the bilinears ??

Define:

$$\boxed{j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x) \quad , \quad j^\mu_5(x) = \bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)}$$

"VECTOR" "AXIAL VECTOR"

VECTOR:

$$\partial_\mu j^\mu = \partial_\mu\bar{\psi}\gamma^\mu\psi + \bar{\psi}\gamma^\mu\partial_\mu\psi$$

{ Recall Dirac Eq: $(i\gamma^\mu\partial_\mu - m)\psi = 0$
$i\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} = 0$ }

$$= (im\bar{\psi})\psi + \bar{\psi}(-im\psi) \quad \Rightarrow$$

$$\boxed{\partial_\mu j^\mu(x) = 0}$$ CONSERVED ✓

AXIAL VECTOR:

$$\partial_\mu j^\mu_5 = (\partial_\mu \bar\psi)\gamma^\mu\gamma^5\psi - \bar\psi\gamma^\mu\gamma^5\partial_\mu\psi$$
$$= (\partial_\mu \bar\psi)\gamma^\mu\gamma^5\psi - \bar\psi\gamma^5\gamma^\mu\partial_\mu\psi$$

$$\Rightarrow \boxed{\partial_\mu j^\mu_5 = 2im\,\bar\psi\psi}$$

CONSERVED AS $m \to 0$

$j^\mu(x)$ and $j^\mu_5(x)$ are Noether currents corresponding to symmetry transformations:

$$\boxed{\psi \to e^{i\alpha}\psi} \qquad \boxed{\psi \to e^{i\alpha\gamma_5}\psi}$$

$U_V(1)$ — symmetry for any m

$U_A(1)$ — symmetry for $m=0$

CHIRAL SYMMETRY

Chiral symmetry is extremely important.
Notice relation to helicity. Relevant when there are massless or nearly massless spin $\frac{1}{2}$ particles.

Fierz Rearrangement

Basic point: products of Dirac bilinears obey interchange relations.

Recall two-component Weyl spinors

$$\sigma^\mu = (1, \sigma^i) \qquad \bar{\sigma}^\mu = (1, -\sigma^i)$$

$$\circledast \quad \boxed{(\sigma^\mu)_{\alpha\dot{\beta}} \, (\sigma_\mu)_{\gamma\dot{\delta}} = 2\,\epsilon_{\alpha\gamma}\,\epsilon_{\dot{\beta}\dot{\delta}}}$$

BASIC RELATION

$$\left\{ \epsilon^{12} = -\epsilon^{21} = \epsilon_{21} = -\epsilon_{12} = 1 \right\}$$

2-d Levi-Civita symbol

α, γ act on $(\frac{1}{2}, 0)$ (ψ_L)

$\dot{\beta}, \dot{\delta}$ act on $(0, \frac{1}{2})$ (ψ_R)

Another useful relation is:

$$\boxed{(\bar{\sigma}^\mu)^{\dot{\alpha}\kappa} \, (\sigma_\mu)_{\gamma\dot{\delta}} = 2\,\delta^{\kappa}{}_{\gamma}\,\delta^{\dot{\alpha}}{}_{\dot{\delta}}}$$

There are many others ...

Let's do example using ⊛.

Consider Dirac Spinors:

$$u_1 = \begin{pmatrix} u_{1L} \\ u_{1R} \end{pmatrix}, \quad u_2 = \begin{pmatrix} u_{2L} \\ u_{2R} \end{pmatrix} \quad \ldots \quad u_n = \begin{pmatrix} u_{nL} \\ u_{nR} \end{pmatrix}$$

Let's ignore dots for now..

$$(\bar{u}_{1R}\, \sigma^\mu\, u_{2R})(\bar{u}_{3R}\, \sigma_\mu\, u_{4R})$$

$$= (\bar{u}_{1R\,\alpha}\, (\sigma^\mu)_{\alpha\beta}\, u_{2R\,\beta})(\bar{u}_{3R\,\gamma}\, (\sigma_\mu)_{\gamma\delta}\, u_{4R\,\delta})$$

$$\{ \text{Use identity ⊛} : (\sigma^\mu)_{\alpha\beta}(\sigma_\mu)_{\gamma\delta} = 2\, \epsilon_{\alpha\gamma}\epsilon_{\beta\delta} \}$$

$$= 2\,(\epsilon_{\alpha\gamma}\, \bar{u}_{1R\,\alpha}\, u_{3R\,\gamma})(\epsilon_{\beta\delta}\, u_{2R\,\beta}\, u_{4R\,\delta})$$

$$\{ \text{Interchange } \beta \text{ and } \delta \}$$

$$= -2\,(\epsilon_{\alpha\gamma}\, \bar{u}_{1R\,\alpha}\, u_{3R\,\gamma})(\epsilon_{\beta\delta}\, u_{2R\,\delta}\, u_{4R\,\beta})$$

$$= -(\bar{u}_{1R}\, \sigma^\mu\, u_{4R})(\bar{u}_{3R}\, \sigma_\mu\, u_{2R})$$

Products of bilinears are antisymmetric under interchange of labels 2 and 4 and also 1 and 3.

Quantization of Dirac Field

Let's construct quantum theory of the Dirac field in analogy to what we did for K-G field.

We constructed $\mathcal{H}$ (schematically):

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \dot{\psi} - \mathcal{L}$$

$$\Rightarrow \boxed{\mathcal{H} = -i \bar{\psi} \gamma_i \nabla_i \psi + m \bar{\psi} \psi}$$

$\Rightarrow$ Dirac Hamiltonian of one-particle Q.M. $\Rightarrow$

$$h_D = -i \gamma^0 \gamma_i \nabla_i + \gamma^0 m$$

$$(= -i \alpha_i \nabla_i + m\beta)$$

(Let's do things the WRONG way...)

Impose : $[\psi_a(x_i), \psi_b^\dagger(y_i)] = \delta^{(3)}(x_i - y_i)\delta_{ab}$

SPINOR INDICES

Now we want representation of these commutation relations in terms of ψ expanded in creation/annihilation operators which diagonalize H, and satisfy Dirac equation.

$$(i\not{\partial} - m)\psi = 0$$

$$(i\gamma_0\partial_0 + i\gamma_i D_i - m)\, u^s(p)\, e^{-ip\cdot x} = 0$$

$$(i\gamma_i D_i - m)\, u^s(p)\, e^{-ip\cdot x} = -i\gamma_0\partial_0\, u^s(p)\, e^{-ip\cdot x}$$

$\otimes -\gamma^0 \Rightarrow \quad h_D\, u^s(p)\, e^{-ip\cdot x} = E_p\, u^s(p)\, e^{-ip\cdot x}$

So $\left\{ \begin{array}{l} u^s(p_i)\, e^{ip_i x_i} \text{ are eigenfunctions of } h_D \text{ w/ eigenvalue } E_p \\ v^s(p_i)\, e^{-ip_i x_i} \quad " \quad " \quad\quad " \quad " \quad " \quad " \quad -E_p \end{array} \right\}$

$\Rightarrow$ complete set of eigenfunctions!

for given $p_i \Rightarrow 2\ u$'s $+ 2\ v$'s $\Rightarrow$
4 eigenfunctions of 4×4 matrix h_D

Expand ψ in this basis in Schrödinger Rep.

$$\psi(x_i) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} e^{ip_i x_i} \sum_{s=1,2} \left[a_p^s u^s(p_i) + b_{-p}^s v^s(-p_i) \right]$$

operator coefficients

Postulate:

$$\underbrace{[a_p^r, a_q^{s+}] = [b_p^r, b_q^{s+}] = (2\pi)^3 \delta^{(3)}(p_i - q_i)\,\delta^{rs}}$$

(else = 0)

$$[\psi(x_i), \psi^+(y_i)] = \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{\sqrt{2E_p}\sqrt{2E_q}} e^{i(p_i x_i - q_i y_i)}$$

$$\times \sum_{r,s} \left([a_p^r, a_q^{s+}]\, u^r(p_i)\bar{u}^s(q_i)\gamma^0 + [b_{-p}^r, b_{-q}^{s+}]\, v^r(p_i)\bar{v}^s(-p_i)\gamma^0 \right)$$

$$\left\{ \text{Recall:} \quad \begin{aligned} \sum_s u^s(p)\bar{u}^s(p) &= \not{p} + m \\ \sum_s v^s(p)\bar{v}^s(p) &= \not{p} - m \end{aligned} \right\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{ip_i(x_i - y_i)} (\gamma^0 p_0 - \gamma^i p_i + m + \gamma^0 p_0 + \gamma^i p_i - m)\gamma^0$$

$$= \int \frac{d^3p}{(2\pi)^3} e^{ip_i(x_i - y_i)}$$

$$\Rightarrow \quad [\psi(x_i), \psi^+(y_i)] = \delta^{(3)}(x_i - y_i)\, \underline{1}_{4\times 4}$$

All looks good so far...

Now let's express H in terms of a and b:

$$H = \int d^3x\, \psi^+ [-i\gamma^0\gamma_i \partial_i + m\gamma^0]\psi$$

$$= \int d^3x \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{\sqrt{2E_p}\sqrt{2E_q}}$$

$$\times e^{-ip_i x_i} \sum_s \left(a_p^{+s} u^{s+}(p_i) + b_{-p}^{s+} v^{s+}(-p_i) \right)$$

$$\times [-i\gamma^0\gamma_i (iq_i) + m\gamma^0)$$

$$\times e^{iq_i x_i} \sum_r \left(a_q^r u^r(q_i) + b_{-q}^r v^r(-q_i) \right)$$

$$= \int d^3x \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{\sqrt{2E_p}\sqrt{2E_q}} e^{-ix_i(p_i - q_i)}$$

$$\sum_{s,r} \Big(a_p^{+s} a_q^r u^{s+}(p_i)\gamma_0[\gamma_i q_i + m) u^r(q_i)$$

$$+ b_{-p}^{s+} b_{-q}^r v^{s+}(-p_i)\gamma_0[\gamma_i q_i + m) v^r(-q_i) \Big)$$

Use Dirac Equation:

$$\gamma^0(\gamma^i p_i + m)u = E_p u \quad \text{etc.}$$

$$H = \int \frac{d^3p}{(2\pi)^3} d^3q \frac{1}{\sqrt{2E_p}\sqrt{2E_q}} \sum_{s,r} \Big(a_p^{+s} a_q^r E_q u^{s+}(p_i) u^r(q_i)$$

$$- b_{-p}^{s+} b_{-q}^r E_q v^{s+}(-p_i) v^r(-q_i) \Big)$$

$$\times \delta^{(3)}(q_i - p_i)$$

Doing q integration and using

$$\left\{ \begin{array}{l} u^{r+}(p)\, u^s(p) = 2E_p \delta^{rs} \\ v^{r+}(p)\, v^s(p) = 2E_p \delta^{rs} \end{array} \right\}$$

$$\Rightarrow \boxed{H = \int \frac{d^3p}{(2\pi)^3} \sum_s \left(E_p a_p^{+s} a_p^s - E_p b_p^{s+} b_p^s \right)}$$

This is bad!!

Unboundedness of H.

By creating particles w/ b^+ we can lower energy indefinitely !!