Let's verify that general solution satisfies the Dirac equation.

$$\begin{pmatrix} -m & i\sigma\cdot\partial \\ i\bar{\sigma}\cdot\partial & -m \end{pmatrix}\begin{pmatrix} u_L \\ u_R \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} -m & \sigma\cdot p \\ \bar{\sigma}\cdot p & -m \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix} = 0$$

$$\boxed{\begin{aligned} -m\sqrt{\sigma\cdot p}\,\xi + \sigma\cdot p\sqrt{\bar{\sigma}\cdot p}\,\xi &= 0 \\ \bar{\sigma}\cdot p\sqrt{\sigma\cdot p}\,\xi - m\sqrt{\bar{\sigma}\cdot p}\,\xi &= 0 \end{aligned}}$$

Consider $-m\sqrt{(\bar{\sigma}\cdot p)(\sigma\cdot p)} + (\sigma\cdot p)(\bar{\sigma}\cdot p) \overset{??}{=} 0$

$$\left\{ \begin{aligned} \bar{\sigma}\cdot p\,\sigma\cdot p &= p_0^2 - \sigma_i\sigma_j\,p_i p_j \\ &= p_0^2 - p_i p_i = p^2 = m^2 \end{aligned} \right\}$$

$$\Rightarrow \bar{\sigma}\cdot p\,\sigma\cdot p = p^2 = m^2$$

$U(p)$ satisfies Dirac equation !!

In practice it is useful to work with specific spinors: Useful choice is <u>eigenstates of σ_3</u>.

$$\sigma_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \sigma_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = - \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\uparrow : \quad u(p) = \begin{pmatrix} \sqrt{E - p_3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \sqrt{E + p_3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \xrightarrow[\text{large BOOST}]{} \sqrt{2E} \begin{pmatrix} 0 \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

$$\downarrow : \quad u(p) = \begin{pmatrix} \sqrt{E + p_3} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \sqrt{E - p_3} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \xrightarrow[\text{large BOOST}]{} \sqrt{2E} \begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 \end{pmatrix}$$

Hence, as $\eta \to \infty$, states become those of massless particle (OBVIOUSLY)

(This is reason for $\sqrt{m}$ convention: keeps expressions <u>finite</u> in $m=0$ limit.)

The 2 massless solutions are eigenstates of the <u>HELICITY</u> operator.

$$\boxed{h = \hat{p}_i S_i = \tfrac{1}{2}\hat{p}_i \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}}$$

Component of spin S in direction of motion

$h = +\frac{1}{2}$ is right-handed

$h = -\frac{1}{2}$ is left-handed

$$\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

(ORIGIN OF NOTATION)

helicity depends on frame of reference, EXCEPT for massless particles.

(The solutions of the WEYL equations are states of definite helicity.)

Here Lorentz invariance is statement that ψ_L and ψ_R transform irreducibly in different representations of the Lorentz group.

Lorentz invariant normalization of $u(p)$?

(Recall $u^\dagger u$ not L.I.)

$$u^\dagger u = \left(\xi^\dagger \sqrt{p\cdot\sigma} \,,\, \xi^\dagger \sqrt{p\cdot\bar\sigma} \right) \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix}$$

$$= \xi^\dagger \xi \left(p\cdot\sigma + p\cdot\bar\sigma \right)$$

$$= 2E_p\, \xi^\dagger \xi$$

Now say $\underline{\bar u(p) = u^\dagger(p)\gamma^0}$

Then,

$$\bar u u = \left(\xi^\dagger \sqrt{p\cdot\sigma} \,,\, \xi^\dagger \sqrt{p\cdot\bar\sigma} \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix}$$

$$= \xi^\dagger \xi\, 2\sqrt{p\cdot\sigma\, p\cdot\bar\sigma}$$

$$\Rightarrow \boxed{\bar u u = 2m\, \xi^\dagger \xi}$$

We can choose $\xi^\dagger \xi = 1$

{ Note: For a massless particle, must use $\underline{u^\dagger u = 2E_p \xi^\dagger \xi}$ }

Summary

General Solutions of the Dirac Equation

+ frequency: $\boxed{\psi(x) = u(p)\, e^{-ip\cdot x} \qquad \underline{p^2 = m^2} \qquad \underline{p^0 > 0}}$

2 linearly independent solutions:

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\; \xi^s \\ \sqrt{p\cdot\bar\sigma}\; \xi^s \end{pmatrix} \qquad \underline{s = 1,2}$$

$\bar u^r(p)\, u^s(p) = 2m\, \delta^{rs}$ or $u^{\dagger r}(p)\, u^s(p) = 2E_p\, \delta^{rs}$

- frequency: $\boxed{\psi(x) = v(p)\, e^{+ip\cdot x} \qquad \underline{p^2 = m^2} \qquad \underline{p^0 > 0}}$

$$v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\; \eta^s \\ -\sqrt{p\cdot\bar\sigma}\; \eta^s \end{pmatrix} \qquad \underline{s = 1,2}$$

$\bar v^r(p)\, v^s(p) = -2m\, \delta^{rs}$ or $v^{\dagger r}(p)\, v^s(p) = +2E_p\, \delta^{rs}$