We know how ψ transforms w/ respect to Lorentz transformations:

$$\boxed{\Lambda_{\frac{1}{2}} = e^{-\frac{i}{2}\omega_{\mu\nu} S^{\mu\nu}}}$$

$\therefore$

$$\boxed{\psi(x) \to \Lambda_{\frac{1}{2}} \psi(\Lambda^{-1} x)}$$

what about the K-G equation?

$$\text{Sq} \qquad \underline{(\Box + m^2)\psi = 0}$$

Lorentz Invariant ??

$$(\Box + m^2)\psi \to (\Box + m^2)\Lambda_{\frac{1}{2}}\psi$$
$$= \Lambda_{\frac{1}{2}}(\Box + m^2)\psi$$
$$= 0 \qquad \text{YES!!}$$

$\Lambda^{\frac{1}{2}}$ is made from γ^μ's which act in an internal 4×4 space!!

But there is a ~~stronger~~ equation which implies the K-G equation!

If we treat γ^μ as an "honest" 4-vector, we can write:

$$\boxed{(i\gamma^\mu \partial_\mu - m)\,\psi(x) = 0}$$

Dirac Equation

Lorentz invariant?

$$(i\gamma^\mu \partial_\mu - m)\,\psi(x) \to [i\gamma^\mu (\Lambda^{-1})^\nu{}_\mu \partial_\nu - m]\,\Lambda_{\frac{1}{2}} \psi(\Lambda^{-1}x)$$

$$= \Lambda_{\frac{1}{2}} [i \underbrace{\Lambda_{\frac{1}{2}}^{-1} \gamma^\mu \Lambda_{\frac{1}{2}} (\Lambda^{-1})^\nu{}_\mu} \partial_\nu - m]\,\psi(\Lambda^{-1}x)$$

$\therefore$ If $\boxed{\Lambda_{\frac{1}{2}}^{-1} \gamma^\mu \Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\alpha \gamma^\alpha}$

{ This says that γ^μ is invariant w/ respect to simultaneous rotation of vector and spinor indices }

Seems reasonable: let's check it.

In infinitesimal form,

$$(\mathbb{1} + \tfrac{i}{2}\omega_{\rho\sigma} S^{\rho\sigma})\gamma^\mu(\mathbb{1} - \tfrac{i}{2}\omega_{\rho\sigma} S^{\rho\sigma}) = (1 - \tfrac{i}{2}\omega_{\rho\sigma} J^{(\rho\sigma)})^\mu{}_\alpha \gamma^\alpha$$

Matching at $O(\omega)$: $[\gamma^\mu, S^{\rho\sigma}] \overset{??}{=} (J^{(\rho\sigma)})^\mu{}_\alpha \gamma^\alpha$

$$= i(\delta^{\rho\mu}\gamma^\sigma - \gamma^\rho\delta^{\sigma\mu})$$

Yes! Check it.

$\therefore$ $(i\gamma^\mu\partial_\mu - m)\psi(x) = 0$

is Lorentz invariant

Now consider

$$(-i\gamma^\nu\partial_\nu - m)(i\gamma^\mu\partial_\mu - m)\psi(x) = 0$$

$$(\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu + m^2)\psi(x) = 0$$

$$\Rightarrow \left[\tfrac{1}{2}\{\gamma^\mu,\gamma^\nu\}\partial_\nu\partial_\mu + m^2\right]\psi(x) = 0$$

$$\Rightarrow (\Box + m^2)\psi(x) = 0 \quad \boxed{\text{B}}$$

K-G Equation

$\therefore$ Dirac Eq. $\Rightarrow$ K-G Eq.

What about a Lagrangian for ψ??

How do we multiply spinors to get a Lorentz scalar??

Say $\psi^\dagger \psi \to \psi^\dagger \Lambda_{\frac{1}{2}}^\dagger \Lambda_{\frac{1}{2}} \psi$

But $\Lambda_{\frac{1}{2}}^\dagger \Lambda_{\frac{1}{2}} \neq \mathbb{1}$ (not unitary) as generators not Hermitian!!

Consider:

$$\boxed{\bar{\psi} \equiv \psi^\dagger \gamma^0} \qquad \left(\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \right)$$

$$\bar{\psi} \to \psi^\dagger \left(\mathbb{1} + \tfrac{i}{2} \omega_{\mu\nu} (S^{\mu\nu})^\dagger \right) \gamma^0$$

sum over $\{\mu, \nu\}$ $\Rightarrow$ 6 non-zero terms:

$\{ij\} \Rightarrow \quad (S^{ij})^\dagger = S^{ij}, \quad [S^{ij}, \gamma^0] = 0$

$\{0\nu\}$, $\{\mu 0\}$ $\Rightarrow \quad (S^{0\nu})^\dagger = -S^{0\nu}, \quad \{S^{0\nu}, \gamma^0\} = 0$

2 minus signs!

$$\Rightarrow \quad \bar{\psi} \to \psi^\dagger \gamma^0 \left(\mathbb{1} + \tfrac{i}{2} \omega_{\mu\nu} S^{\mu\nu} \right)$$
$$= \bar{\psi} \Lambda_{\frac{1}{2}}^{-1}$$

$\therefore \quad \bar{\psi}\psi \to \bar{\psi}\psi$ is a Lorentz scalar

Now we can write:

$$\boxed{\mathcal{L} = \bar{\psi} i \gamma^\mu \partial_\mu \psi - m \bar{\psi}\psi}$$

Check Lorentz invariance of kinetic term:

$$\bar{\psi} i \gamma^\mu \partial_\mu \psi \to \bar{\psi} \Lambda_{\frac{1}{2}}^{-1} i \gamma^\mu (\Lambda^{-1})_\mu{}^\nu \partial_\nu \Lambda_{\frac{1}{2}} \psi$$
$$= \bar{\psi} i [\underbrace{\Lambda_{\frac{1}{2}}^{-1} \gamma^\mu \Lambda_{\frac{1}{2}} (\Lambda^{-1})_\mu{}^\nu}_{= \gamma^\nu}] \partial_\nu \psi \quad \checkmark$$

Euler-Lagrange Equation

First write:

$$\mathcal{L} = \frac{i}{2} \left[\bar{\psi} \gamma^\mu (\partial_\mu \psi) - (\partial_\mu \bar{\psi}) \gamma^\mu \psi \right] - m \bar{\psi}\psi$$

(Treat $\psi, \bar{\psi}$ as independent fields.)

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})}\right) - \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = 0$$

$$-\frac{i}{2}\gamma^\mu\partial_\mu\psi + m\psi - \frac{i}{2}\gamma^\mu\partial_\mu\psi = 0$$

$$\Updownarrow$$

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \quad \checkmark$$

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\right) - \frac{\partial \mathcal{L}}{\partial \psi} = 0$$

$$\frac{i}{2}\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} + \frac{i}{2}\partial_\mu\bar{\psi}\gamma^\mu = 0$$

$$\Updownarrow$$

$$[(i\gamma^\mu\partial_\mu - m)\psi]^\dagger = 0 \quad \checkmark$$

More on Spin $\frac{1}{2}$ Representations of the Lorentz Group: <u>WEYL SPINORS</u>

<u>Recall</u>:

$$\psi \to \Lambda_{\frac{1}{2}}\psi = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}$$

$$S^{0i} = -\frac{i}{2}\begin{pmatrix}\sigma^i & 0 \\ 0 & -\sigma^i\end{pmatrix}, \qquad S^{ij} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix}\sigma^k & 0 \\ 0 & \sigma^k\end{pmatrix}$$

"BOOSTS" "ROTATIONS"

Consider infinitesimal form:

$$\psi \to \left(\mathbb{1}_{4\times4} - i\left[\omega_{0i} S^{0i} + \omega_{ij} S^{ij}\right]\right)\psi$$

$$\to \begin{pmatrix} \mathbb{1}_{2\times2} - \omega_{0i}\frac{\sigma^i}{2} - i\omega_{ij}\epsilon^{ijk}\frac{\sigma^k}{2} & 0 \\ 0 & \mathbb{1}_{2\times2} + \omega_{0i}\frac{\sigma^i}{2} - i\omega_{ij}\epsilon^{ijk}\frac{\sigma^k}{2} \end{pmatrix} \times \psi$$

Say $\omega_{0i} = \beta_i$ $\quad \omega_{ij}\epsilon^{ijk} = \theta^k$

Then with

$$\boxed{\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}} \Rightarrow$$ Left/right handed $\boxed{\text{WEYL spinors}}$

$$\psi_L \to \left(\mathbb{1}_{2\times2} - \beta_i\frac{\sigma^i}{2} - i\theta_k\frac{\sigma^k}{2}\right)\psi_L$$

$$\psi_R \to \left(\mathbb{1}_{2\times2} + \beta_i\frac{\sigma^i}{2} - i\theta_k\frac{\sigma^k}{2}\right)\psi_R$$

OR

$$\boxed{\begin{aligned} \psi_L &\to \left(\mathbb{1} - i\theta_i J_i + \overline{\omega}_i K_i\right)\psi_L \\ \psi_R &\to \left(\mathbb{1} - i\theta_i J_i - \overline{\omega}_i K_i\right)\psi_R \end{aligned}}$$

Recall linear combinations of J_i and K_i that made GROUP structure manifest:

$$A_i = \tfrac{1}{2}(J_i + iK_i)$$

$$B_i = \tfrac{1}{2}(J_i - iK_i)$$

$$\psi_L \to \left(\mathbb{1} - i\theta_i (2A_i)\right)\psi_L$$

$$\boxed{\psi_L \in (\tfrac{1}{2}, 0) \text{ of } SU_L(2) \otimes SU_R(2)}$$

$$\psi_R \to \left(\mathbb{1} - i\theta_i (2B_i)\right)\psi_R$$

$$\boxed{\psi_R \in (0, \tfrac{1}{2}) \text{ of } SU_L(2) \otimes SU_R(2)}$$

$$\therefore \quad \boxed{\psi \in (\tfrac{1}{2}, 0) \oplus (0, \tfrac{1}{2})}$$

Reducible representation of Lorentz group $SU(2) \otimes SU(2)$

(This group structure is manifest in the chiral representation of the γ_μ's)

Dirac Equation(s) for Weyl spinors ??

$$(i\gamma^\mu \partial_\mu - m)\psi = \begin{pmatrix} -m & i(\partial_0 + \sigma_i \nabla_i) \\ i(\partial_0 - \sigma_i \nabla_i) & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

The two representations ψ_L and ψ_R are mixed by the mass term,

So $\underline{m = 0}$ $\Rightarrow$

$$\boxed{\begin{aligned} i(\partial_0 + \sigma_i \nabla_i)\psi_R &= 0 \\ i(\partial_0 - \sigma_i \nabla_i)\psi_L &= 0 \end{aligned}}$$

Weyl Equations

$\left\{ \begin{array}{l} \text{CHIRALITY exhibited by Weyl spinors} \\ \text{important for discussion of neutrinos} \\ \text{(+ other massless fermions) AND for} \\ \text{SUSY} \end{array} \right\}$

Free Solutions of the Dirac Equation

First, some notation.

4-d Weyl notation

$$\sigma^\mu = (\mathbb{1}, \sigma^i) \qquad \bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i)$$

(bar has nothing to do w/ $\bar{\psi}$)

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

Dirac Eqn. $\Rightarrow$ $\begin{pmatrix} -m & i\sigma\cdot\partial \\ i\bar{\sigma}\cdot\partial & -m \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$

Weyl Eqns $\Rightarrow$

$$\boxed{\begin{aligned} i\bar{\sigma}\cdot\partial\psi_L &= 0 \\ i\sigma\cdot\partial\psi_R &= 0 \end{aligned}}$$

As ψ satisfies K-G equation, it can be written as sum of plane waves:

$$\psi(x) = u(p)\, e^{-ip\cdot x} \qquad w/ \quad p^2 = m^2$$

$$(\Box + m^2)\psi = (-p^2 + m^2)\psi = 0 \quad ✓$$

Say $p_0 > 0$ (positive frequency)

ψ must also satisfy Dirac Equation.

$$[\, i\gamma^\mu \partial_\mu - m\,]\,\psi(x) = 0$$

$$\Downarrow$$

$$[\,\gamma^\mu p_\mu - m)\, u(p) = 0$$

Let's analyze in the REST FRAME $p = p_0 = (m, \vec{0})$

$$(\gamma^0 p_0 - m)\, u = 0$$

$$\left[\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} m - m\right] u = m \begin{pmatrix} -\mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix} u(p_0) = 0$$

Can write solution as:

$$U(p_0) = \sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}$$

convention

Any 2-component spinor

By convention, assume $\underline{\xi^\dagger \xi = 1}$

How does ξ transform under ROTATIONS?

$$\xi \to \left(1 - i\,\theta_k \frac{\sigma^k}{2}\right)\xi$$

transforms as 2-component spinor of the ROTATION group.

Hence, ξ determines the spin orientation of the spin $\frac{1}{2}$ field. E.g., if

$\xi = \begin{pmatrix}1\\0\end{pmatrix} \Rightarrow$ spin up in z-direction

NOTE: Dirac equation implies that only <u>two</u> of the <u>four</u> components of ψ can be chosen.

Now that we have solution $U(p)$ in the rest frame, we can obtain solution in any frame by <u>boosting</u>.

Consider boost in 3-direction:

$$\boxed{\begin{aligned} p_3' &= \gamma(p_3 + \beta E) & E' &= \gamma(E + \beta p_3) \\ &= \gamma\beta m & &= \gamma m \\ & (p_3 = 0,\ E = m) \end{aligned}}$$

Hence,

$$\begin{pmatrix} E \\ p_3 \end{pmatrix}' = \gamma m \begin{pmatrix} 1 \\ \beta \end{pmatrix} = \begin{pmatrix} 1 & \eta \\ \eta & 1 \end{pmatrix} \begin{pmatrix} m \\ 0 \end{pmatrix}$$

(for $\beta \sim \eta$ infinitesimal i.e. $\gamma \sim 1$)

$$\Rightarrow \begin{pmatrix} E \\ p_3 \end{pmatrix}' = \left[\mathbb{1} + \eta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \begin{pmatrix} m \\ 0 \end{pmatrix}$$

$$\left\{ \text{Could have gotten directly from: } \quad p^{\alpha\prime} = \left(\delta^\alpha_\beta - i\, \omega_{0i} \, (j^{0i})^\alpha{}_\beta \right) p^\beta \right\}$$

For "finite" η we have:

$$\begin{pmatrix} E \\ p_3 \end{pmatrix}' = e^{\eta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \begin{pmatrix} m \\ 0 \end{pmatrix}$$

$$= \left[\cosh\eta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sinh\eta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \begin{pmatrix} m \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix} \begin{pmatrix} m \\ 0 \end{pmatrix} = \begin{pmatrix} m\cosh\eta \\ m\sinh\eta \end{pmatrix}$$

$$\therefore \quad E \pm p_3 = m e^{\pm\eta}$$

Now apply same boost to $u(p_0)$:

$$\underline{u(p) = \Lambda_{\frac{1}{2}}^{B_3}\, u(p_0)}$$

$$= e^{-i\omega_{03} S^{03}}\, u(p_0)$$

$$= \exp\left[-\tfrac{1}{2}\eta \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix} \right] \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$\left\{ \text{Rem:} \quad e^{-i\theta\hat{A}} = \mathbb{1}\cos\theta - i\hat{A}\sin\theta \qquad \eta = i\theta \right\}$$

$$= \left[\cosh(\tfrac{1}{2}\eta) \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} - \sinh(\tfrac{1}{2}\eta) \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix} \right] \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$= \left[\begin{pmatrix} \frac{(e^{\eta/2}+e^{-\eta/2})}{2}\mathbb{1} & 0 \\ 0 & \frac{(e^{\eta/2}+e^{-\eta/2})}{2}\mathbb{1} \end{pmatrix} - \begin{pmatrix} \frac{(e^{\eta/2}-e^{-\eta/2})}{2}\sigma^3 & 0 \\ 0 & -\frac{(e^{\eta/2}-e^{-\eta/2})}{2}\sigma^3 \end{pmatrix} \right] \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$= \begin{pmatrix} e^{\eta/2}\frac{(\mathbb{1}-\sigma^3)}{2} + e^{-\eta/2}\frac{(\mathbb{1}+\sigma^3)}{2} & 0 \\ 0 & e^{\eta/2}\frac{(\mathbb{1}+\sigma^3)}{2} + e^{-\eta/2}\frac{(\mathbb{1}-\sigma^3)}{2} \end{pmatrix} \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$\left\{ \text{Now recall that} \quad \underline{\sqrt{E \pm p_3} = \sqrt{m}\, e^{\pm \eta/2}} \right\}$$

$$= \begin{pmatrix} \left[\sqrt{E+p_3}\,\frac{(\mathbb{1}-\sigma^3)}{2} + \sqrt{E-p_3}\,\frac{(\mathbb{1}+\sigma^3)}{2} \right] \xi \\ \left[\sqrt{E+p_3}\,\frac{(\mathbb{1}+\sigma^3)}{2} + \sqrt{E-p_3}\,\frac{(\mathbb{1}-\sigma^3)}{2} \right] \xi \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{(E+p_3)\frac{(\mathbb{1}-\sigma^3)}{2} + (E-p_3)\frac{(\mathbb{1}+\sigma^3)}{2}}\; \xi \\ \sqrt{(E+p_3)\frac{(\mathbb{1}+\sigma^3)}{2} + (E-p_3)\frac{(\mathbb{1}-\sigma^3)}{2}}\; \xi \end{pmatrix}$$

$$\left\{ \underline{\text{Note}}: \quad P_{\pm} = \frac{(\mathbb{1} \pm \sigma_3)}{2} \qquad \begin{matrix} P_{\pm}^2 = P_{\pm} \\ P_+ P_- = 0 \end{matrix} \right\}$$

$$= \begin{pmatrix} \sqrt{E + \sigma_3 p_3}\ \xi \\ \sqrt{E - \sigma_3 p_3}\ \xi \end{pmatrix}$$

$$\Rightarrow \quad \boxed{U(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\ \xi \\ \sqrt{p \cdot \bar{\sigma}}\ \xi \end{pmatrix}}$$

(Understood that take $+\sqrt{\ }$ ")

Valid for arbitrary p !!