Lorentz Invariance in Wave Equations

What does it mean for a wave equation to be "relativistically invariant" ??

Lagrangian formulation of field theory makes it easy to discuss Lorentz invariance:

$\mathcal{L}$ is a scalar.

{ If boosts leave $\mathcal{L}$ unchanged, boosts of extremum will be another extremum. Hence E.O.M. will be unchanged!! }

Consider K-G field.

$$\boxed{x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu}$$

Lorentz transformation

4x4 matrix

Say $x = \Lambda x_0$

$$\varphi(x) \to \varphi'(x) = \varphi(\Lambda^{-1} x)$$

{ transformed field evaluated at boosted point, x, is same as original field at value before boosting. }

Recall $\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2} m^2 \varphi^2$

φ^2 transforms as φ (only argument transforms)

$$\partial_\mu \varphi(x) \longrightarrow \underset{\substack{\uparrow \\ x}}{\partial_\mu} (\varphi'(\underset{\substack{\uparrow \\ x}}{x})) = \underset{\substack{\uparrow \\ x_0}}{(\Lambda^{-1})^\nu{}_\mu} (\partial_\nu \varphi)(\underset{\substack{\uparrow \\ x_0}}{\Lambda^{-1} x})$$

$$\left\{ \begin{array}{l} \text{As } g^{\mu\nu} \text{ is Lorentz Invariant,} \\ \qquad (\Lambda^{-1})^\rho{}_\mu (\Lambda^{-1})^\sigma{}_\nu \, g^{\mu\nu} = g^{\rho\sigma} \\ \qquad \text{w/ } (\Lambda^{-1})^\mu{}_\nu = \Lambda_\nu{}^\mu \end{array} \right\}$$

$\therefore$

$$(\partial_\mu \varphi(x))^2 \to g^{\mu\nu} (\partial_\mu(\varphi'(x)))(\partial_\nu(\varphi'(x)))$$

$$= g^{\mu\nu} [(\Lambda^{-1})^\rho{}_\mu \partial_\rho \varphi][(\Lambda^{-1})^\sigma{}_\nu \partial_\sigma \varphi](\Lambda^{-1}x)$$

$$= g^{\rho\sigma} (\partial_\rho \varphi)(\partial_\sigma \varphi)(\Lambda^{-1}x)$$

$$= (\partial_\mu \varphi(\Lambda^{-1}x))^2$$

So $\boxed{\mathcal{L}(x) \longrightarrow \mathcal{L}(\Lambda^{-1}x)}$

$\therefore$ $\mathcal{L}$ is a SCALAR

The action S is a scalar

$$S = \int d^4x \, \mathcal{L}(x)$$

Straightforward to show that E.O.M is invariant.

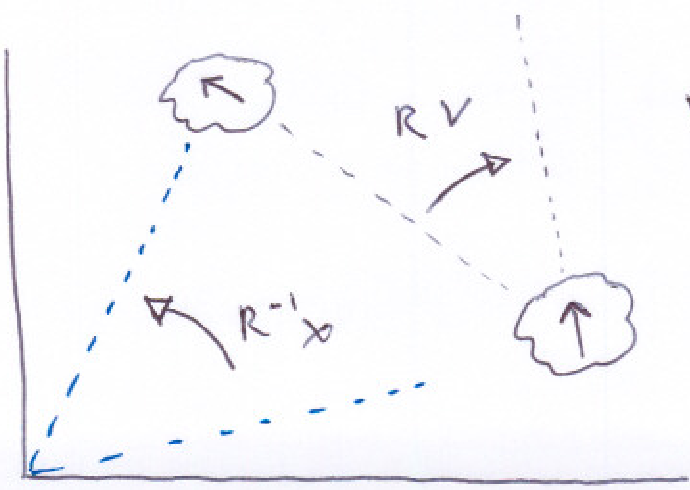
Transformation law for a scalar is simplest possible law for a field.

What about vector field $V^\mu(x)$??

(like E+M vector potential)

$$\boxed{V^\mu(x) \to \Lambda^\mu{}_\nu V^\nu(\Lambda^{-1}x)}$$

e.g. 3-d rotations



$$V^i(x) \to R^{ij} V^j(R^{-1}x)$$

Can do same for <u>tensor</u> fields

$$T^{\mu\nu} \quad \text{and} \quad M^{\mu\nu\cdots\sigma\rho}$$

$\Rightarrow$ Large class of Lorentz invariant equations

But there are still more ....

Consider linear transformation on N-component field:

$$\Phi_a(x) \qquad a = 1, \ldots, N$$

Under Lorentz transformation,

$$\Phi_a(x) \longrightarrow \underbrace{M_{ab}(\Lambda)}_{N \times N \text{ matrix}} \Phi_b(\Lambda^{-1}x)$$

Suppress for a moment the change in field argument: $\Phi \longrightarrow M(\Lambda)\,\Phi$

(*) What are possible allowed forms for $M(\Lambda)$??

Consider two successive transformations:

$$\underline{\Phi} \to M(\Lambda')\, M(\Lambda)\, \underline{\Phi} = M(\Lambda'')\, \underline{\Phi}$$

GROUP PROPERTY

for $\Lambda'' = \Lambda' \Lambda$

The M form N-dimensional REPRESENTATION of the Lorentz group ($SU(2) \otimes SU(2)$)

Rephrase (*):

what are the finite-dimensional matrix representations of the Lorentz group?

First consider the subgroup of rotations

Remember: with spin quantum # s

$$N = 2s + 1$$

(e.g. If $s = \frac{1}{2}$ ($\uparrow$ and $\downarrow$) $N = 2$ ✓)

Let's consider $\underline{\underline{s = \tfrac{1}{2}}}$ in some more detail;

Matrices of this representation are:

$$\boxed{U = e^{-i\theta_i\sigma_i/2}}$$

(in $SU(2)$)
(Rem: $\det = 1$)

θ_i: 3 arbitrary parameters

σ_i: Pauli matrices

Lie algebra ?? "General" Rotations

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

$$R = e^{-i\theta_i J_i}$$

Similarly for Boosts

$$[K_i, K_j] = -i\epsilon_{ijk} J_k$$

etc.

$$B = e^{-i\theta_i K_i}$$

How do we put J_i and K_i together in relativistic notation ??

Recall that J_i is <u>ANGULAR MOMENTUM</u>

$$J_i = \epsilon_{ijk} x_j p_k = -i \epsilon_{ijk} x_j \nabla_k$$

(or $\vec{J} = \vec{x} \times \vec{p}$)

Define $J_{mn} \equiv \epsilon_{mni} J_i = -i \epsilon_{mni} \epsilon_{ijk} x_j \nabla_k$

$$= -i (\delta_{mj} \delta_{nk} - \delta_{mk} \delta_{nj}) x_j \nabla_k$$

$$\Rightarrow \quad J_{mn} = -i (x_m \nabla_n - x_n \nabla_m)$$

Now generalize to 4-d:

$$\boxed{J^{\mu\nu} = i (x^\mu \partial^\nu - x^\nu \partial^\mu)}$$

$$\boxed{[J^{\mu\nu}, J^{\rho\sigma}] = i (g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho})}$$

$\to$ (pair + swapping)

<u>Algebra</u> of (homogeneous) <u>Lorentz group</u>

Can check:

$$\boxed{J_{\mu\nu} = \begin{cases} J_{ij} = -J_{ji} = \epsilon_{ijk} J_k \\ J_{i0} = -J_{0i} = -K_i \end{cases}}$$

Any matrices that satisfy Algebra are representation of the Lorentz group.

E.g.

$$(J^{\mu\nu})_{\alpha\beta} = i\left(\delta^{\mu}_{\alpha}\,\delta^{\nu}_{\beta} - \delta^{\mu}_{\beta}\,\delta^{\nu}_{\alpha}\right)$$

6 matrices

components of matrices

(like $(\sigma^i)_{\alpha\beta}$ in $SU(2)$)

You can verify that J satisfies the Algebra !!

Say $V^\alpha \to \Lambda^\alpha{}_\beta V^\beta$ (V^α - 4 vector)

$$\Lambda^\alpha{}_\beta V^\beta = \left(\delta^\alpha_\beta - \tfrac{i}{2}\,\omega_{\mu\nu}\,(J^{\mu\nu})^\alpha{}_\beta\right)V^\beta$$

infinitesimal, anti-symmetric tensor

Now say $\underline{\omega_{12} = -\omega_{21} = \theta}$ (else $= 0$)

$$\Rightarrow V \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\theta & 0 \\ 0 & \theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} V$$

(rotation in x-y plane)

Now say $\omega_{01} = -\omega_{10} = \beta$

$$\Rightarrow V \longrightarrow \begin{pmatrix} 1 & \beta & 0 & 0 \\ \beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} V \qquad (\text{Boost in } x\text{-direction})$$

For homogeneous Lorentz group we have considered Boosts and Rotations:

$$\boxed{x^{\mu\prime} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu}}$$

now include time + space translations

inhomogeneous Lorentz group

or

Poincare' Group

$P_\mu = i \dfrac{\partial}{\partial x^\mu}$ Energy - Momentum

Adds 4 generators to Lorentz group:

10 total generators

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho})$$

$$[P_\mu, J_{\rho\sigma}] = i(g_{\mu\rho}P_\sigma - g_{\mu\sigma}P_\rho)$$

$$[P_\mu, P_\nu] = 0$$

Algebra of Poincaré group

→ J_i and P_i commute w/ $H = P_0$ but K_i does not!

→ translations in different directions commute.

In some sense a discussion of the representations of the Poincaré group is a logical starting point for Q.F.T.

The Dirac Equations

Rotations: $R = e^{-i\theta_i J_i}$

Boosts: $B = e^{-i\beta_i K_i}$

$$\boxed{\Lambda = e^{-\frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}}}$$

$SU(2) \times SU(2)$
(or $SO(3,1)$)

We are interested in representations of the Poincaré group corresponding to fields of spin $\frac{1}{2}$.

Dirac's trick:

Suppose we have $n \times n$ matrix γ^μ which satisfies:

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbb{1}}$$

$n \times n$

(note: n is arbitrary - not necessarily S-T dimension)

Now define:

$$\boxed{S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]}$$

Commutator ??

Can show:

$$[S^{\mu\nu}, S^{\rho\sigma}] = i(g^{\nu\rho}S^{\mu\sigma} - S^{\nu\sigma}g^{\mu\rho} - g^{\nu\sigma}S^{\mu\rho} + g^{\mu\sigma}S^{\nu\rho})$$

Satisfies Lorentz group algebra!!
(for any n)

Consider $n = d = 3$ Euclidean space

$$g^{\mu\nu} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$

So $\underline{\gamma^k = i\sigma^k}$

CONVENTION!

$$\{\gamma^k, \gamma^l\} = -\{\sigma^k, \sigma^l\} = -2\delta^{kl} \checkmark$$

Then $S^{ij} = -\frac{i}{4}[\sigma^i, \sigma^j] = \frac{1}{2}\underline{\epsilon^{ijk}\sigma^k}$

Matching $\Rightarrow$ $J^k = \frac{\sigma^k}{2}$ 2-d representation of rotation group $SU(2)$

Now consider a representation of γ^μ for $d=4$ Minkowski space.

$n = ??$ must be at least $n=4$ i.e 4×4

Only need 1 explicit representation as all others are unitarily equivalent.

Weyl or Chiral Representation

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

2x2 blocks

E.g. $\gamma^0\gamma^i = \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$ $\gamma^i\gamma^0 = \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$

$\{\gamma^0, \gamma^i\} = 0$

$$\left\{ \gamma^0\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \qquad \{\gamma^0,\gamma^0\} = 2\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \right\}$$

Recall that:

$$K^i = S^{0i} = \frac{i}{4}[\gamma^0,\gamma^i] = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$$

(Not Hermitian, boosts not unitary)

$$S^{ij} = \frac{i}{4}[\gamma^i,\gamma^j] = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \equiv \frac{1}{2}\epsilon^{ijk}\Sigma^k$$

$$\boxed{J^i = \frac{\Sigma^i}{2} = \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}}$$

Note:

$$A^i = \frac{1}{2}(J^i + iK^i) = \frac{1}{4}\begin{pmatrix} 2\sigma^i & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & 0 \end{pmatrix}$$

$$B^i = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \sigma^i \end{pmatrix}$$

$$[A,A] = A$$
$$[B,B] = B$$
$$[B,A] = 0$$

two indep't $SU(2)$'s

i.e. $SU(2)\otimes SU(2)$

A 4-component field ψ that transforms in this representation (w/ K^i and J^i) is called a <u>Dirac spinor</u>