Alternate derivation:

Fourier Transform of D_R:

$$D_R(x-y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)}\, \tilde{D}_R(p)$$

$$(\Box + m^2)\, D_R(x-y) = \int \frac{d^4p}{(2\pi)^4}\, (-p^2 + m^2)\, \tilde{D}_R(p)\, e^{-ip\cdot(x-y)}$$

$$= -i\,\delta^{(4)}(x-y) \qquad \left(\text{shown previously}\right)$$

$$\Rightarrow \quad (-p^2+m^2)\,\tilde{D}_R(p) = -i$$

$$\boxed{\tilde{D}_R(p) = \frac{i}{p^2-m^2}}$$

or $$\boxed{D_R(x-y) = \int \frac{d^4p}{(2\pi)^4}\, \frac{i}{p^2-m^2}\, e^{-ip\cdot(x-y)}}$$

FUNDAMENTAL OBJECT IN QFT !!

$$\boxed{\text{In general, can be evaluated by 4 different contours. } 4^{\circ}}$$

$$\frac{1}{(p_0 \pm i\epsilon)^2 - p_i p_i - m^2}$$

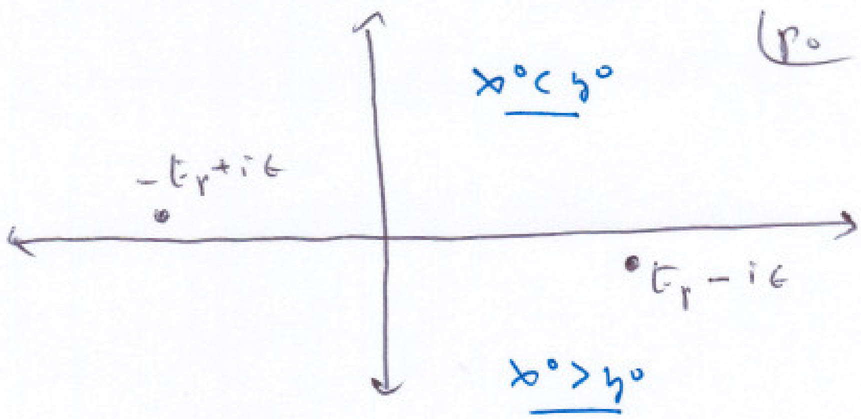
$+$: Retarded

$-$: Advanced

$$\frac{1}{(p^2 - m^2 \pm i\epsilon)}$$

$+$: Feynman

The Feynman Prescription



poles at: $p^0 = \pm (E_p - i\epsilon)$

$$\frac{1}{(p^0 + (E_p - i\epsilon))(p^0 - (E_p - i\epsilon))} = \frac{1}{p_0^2 - (E_p - i\epsilon)^2}$$

$$= \frac{1}{p_0^2 - E_p^2 + i\tilde{\epsilon}} = \frac{1}{p^2 - m^2 + i\epsilon}$$

$x^0 > y^0$ close contour below

$x^0 < y^0$ close contour above

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\epsilon} e^{-ip\cdot(x-y)}$$

PARTICLE MOVING FROM:

$$= \begin{cases} D(x-y) & x^0 > y^0 \\ D(y-x) & x^0 < y^0 \end{cases}$$

x^i to y^i

y^i to x^i (ANTIPARTICLE)

So

$$D_F(x-y) = \Theta(x^0-y^0)\langle 0|\varphi(x)\varphi(y)|0\rangle + \Theta(y^0-x^0)\langle 0|\varphi(y)\varphi(x)|0\rangle$$

$$\Rightarrow \boxed{D_F(x-y) = \langle 0|T\left(\varphi(x)\varphi(y)\right)|0\rangle}$$

Feynman propagator

time-ordering symbol

$T(\ldots)$: PLACE OPERATORS IN ORDER w/ LATEST TO THE LEFT

(Can verify that D_F is Green's fn. by applying $(\Box+m^2)$ ...)

{ The Feynman propagator will appear in Feynman diagrams!

To see utility, we have to consider <u>interactions</u>. }

There is one type of interaction that we can consider given the technology that we've developed:

<u>K-G FIELD COUPLED TO EXTERNAL CLASSICAL SOURCE $j(x)$</u>

$$\boxed{(\Box + m^2)\,\varphi(x) = j(x)}$$

(<u>Rem</u>: $\vec{\nabla}\cdot\vec{E} = \rho$ Coulomb's Law)

<u>Lagrangian</u>?? Recall:

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right) = \frac{\partial\mathcal{L}}{\partial\varphi}$$

<u>E.O.M</u>

Say $\boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m^2\varphi^2 + \underline{j(x)\varphi(x)}}$

Assume $\underline{\underline{j(x) \neq 0}}$ for time interval Δt

If initial state is the vacuum, what is final state after $j(x)$ has been turned on and off??

<u>Before turn on:</u>

$$\varphi_0(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x} \right)$$

Without $j(x)$ this would be solution for all time.

{ Let's use fact that $D_R(x)$ is a <u>Green's fn.</u> to construct general solution. }

Say $\varphi(x) = \varphi_0(x) + i\int d^4y \, D_R(x-y)\, j(y)$

{ <u>check:</u>

$(\Box + m^2)\varphi(x) = 0 + i\int d^4y \left(-i\delta^{(4)}(x-y)\right) j(y)$

$= j(x)$ }

Evaluate explicitly:

$$\varphi(x) = \varphi_0(x) + i\int d^4y\, \Theta(x^0 - y^0)\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p} \times\left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right) j(y)$$

{ Wait until all $j(y)$ is in the past $\Rightarrow$ $\Theta(x^0 - y^0) = 1$ over all domain of integration. }

$$\boxed{\varphi(x) = \varphi_0(x) + i\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\left(e^{-ip\cdot x}\tilde{j}(p) - e^{ip\cdot x}\tilde{j}^*(p)\right)}$$

$$\tilde{j}(p) \equiv \int d^4y\, e^{ip\cdot y} j(y) \qquad (\text{w/ } p^2 = m^2)$$

Grouping terms by exponentials $\Rightarrow$

$$\varphi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left[\left(a_p + i\frac{\tilde{j}(p)}{\sqrt{2E_p}}\right)e^{-ip\cdot x} + \text{h.c.}\right]$$

Kind of "minimal substitution":

$$\boxed{a_p \rightarrow a_p + i\frac{\tilde{j}(p)}{\sqrt{2E_p}}}$$

Also have:

$$H = \int \frac{d^3p}{(2\pi)^3}\, E_p \left(a_p^\dagger - \frac{i}{\sqrt{2E_p}} \tilde{j}^*(p) \right) \left(a_p + \frac{i}{\sqrt{2E_p}} \tilde{j}(p) \right)$$

$\therefore$ Energy of the system *after* source has been turned off is:

$$\langle 0 | H | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} |\tilde{j}(p)|^2$$

G.S. of free theory

Interpret:

$\frac{|\tilde{j}(p)|^2}{2E_p}$: PROBABILITY DENSITY FOR CREATING PARTICLE IN MODE p.

total # of particles produced:

$$\int dN = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |\tilde{j}(p)|^2$$

{ Only Fourier components of $j(x)$ in resonance w/ $p^2 = m^2$ (on-shell) K-G waves contribute. }

(Before moving on to fermions, ...)

The Lorentz Group

Review of rotations

$$\boxed{\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R \begin{pmatrix} x \\ y \\ z \end{pmatrix}}$$

(R is rotation matrix)

Component notation: $x^{i\prime} = R^i{}_j \, x^j$

Rotations preserve distance from the origin:

$$x^{i\prime} x_i' = x^j x_j$$

$$R^i{}_k R^l{}_i \, x^k x_l = x^j x_j$$

$$\Rightarrow \boxed{R^i{}_k R^l{}_i = \delta_k{}^l \quad \text{or} \quad R^T R = \mathbb{1}}$$

R is orthogonal 3×3 matrix

These matrices form a GROUP : $O(3)$

Group properties

(Are they satisfied for rotations?)

1) If R_1 and R_2 are rotations, so is
$$R_1 \cdot R_2 \quad ✓$$

2) $\mathbb{1} \cdot R = R \cdot \mathbb{1}$ for each R ✓

3) Every R has an R^{-1} such that
$$R \cdot R^{-1} = R^{-1} R = \mathbb{1} \quad ✓$$

4) For every R_1, R_2, R_3
$$(R_1 \cdot R_2) \cdot R_3 = R_1 \cdot (R_2 \cdot R_3) \quad ✓$$

Consider rotations about z-axis:

$$R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Similarly,

$$R_y(\psi) = \begin{pmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{pmatrix} \quad + \quad R_x(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{pmatrix}$$

$$[R_x(\varphi), R_z(\theta)] \neq 0$$

$\Rightarrow$ $O(3)$ is non-abelian Lie group

(continuous group w/ ∞ # of elements)

General rotation in 3-d has 3 parameters: θ, φ, ψ (Rem: Euler angles)

(R has 9 elements; $R^T R = \mathbb{1} \Rightarrow$ 6 conditions)

Corresponding to 3 parameters are 3 GENERATORS defined by:

$$J_z = \frac{1}{i} \frac{d}{d\theta} R_z(\theta) \Big|_{\theta=0} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$J_x = \frac{1}{i} \frac{d}{d\theta} R_x(\varphi) \Big|_{\varphi=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$J_y = \frac{1}{i} \frac{d}{d\theta} R_y(\psi) \Big|_{\psi=0} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

<u>Note</u>: The J_i are Hermitian:

$$J_i = J_i^\dagger$$

$$R_z(\delta\theta) = \mathbb{1} + i J_z \delta\theta + \dots$$ <u>etc.</u>

Can easily show:

$$\boxed{[J_i, J_j] = i \epsilon_{ijk} J_k}$$

$(\hbar)$

<u>Angular</u> <u>Momentum</u>

What about <u>finite</u> rotation?

$$R_z(\theta) = \lim_{N\to\infty} (\mathbb{1} + i J_z \delta\theta)^N = \lim_{N\to\infty} \left(\mathbb{1} + i J_z \frac{\theta}{N}\right)^N$$

$$\text{"} \quad = e^{i J_z \theta}$$

In general,

$$\boxed{R_n(\theta) = e^{-i J_i n_i \theta} = e^{-i J_i \theta_i} \qquad \theta_i \equiv n_i \theta}$$

(<u>Recall</u>: 2 to 1, $SU(2) \to O(3)$)

Recall <u>BOOSTS</u>!

Now relative motion along common x-axis.

$$\boxed{\begin{array}{cc} x^{0\prime} = \gamma(x^0 + \beta x^1) & x^{1\prime} = \gamma(\beta x^0 + x^1) \\ x^{2\prime} = x^2 & x^{3\prime} = x^3 \end{array}}$$

$$\left\{ \gamma = (1-\beta^2)^{-\frac{1}{2}} \qquad \beta = v/c \right\}$$

As $\gamma^2 - \beta^2\gamma^2 = 1$, we may parametrize

$$\boxed{\begin{array}{cc} \gamma = \cosh\varphi & \gamma\beta = \sinh\varphi \\ \multicolumn{2}{c}{\tanh\varphi = \beta} \end{array}}$$

$$\begin{pmatrix} x^{0\prime} \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{pmatrix} = \underbrace{\begin{pmatrix} \cosh\varphi & \sinh\varphi & 0 & 0 \\ \sinh\varphi & \cosh\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{BOOST MATRIX: } \mathcal{B}} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

In analogy with rotations,

$$\boxed{K_x = -i \left.\frac{\partial B}{\partial \varphi}\right|_{\varphi=0} = -i \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_y = -i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_z = -i \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}}$$

K_i are the BOOST generators

In 4x4 notation ROTATION GENERATORS ARE

$$\boxed{J_x = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad J_y = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad J_z = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}$$

Most general Lorentz transformation:

BOOSTS in 3 directions
ROTATIONS in/about 3 axes

6 generators, 6 parameter "GROUP"

One may easily verify that

$$\boxed{\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk} J_k \\ [J_i, K_j] &= i\epsilon_{ijk} K_k \\ [K_i, K_j] &= -i\epsilon_{ijk} J_k \end{aligned}}$$

Hence, pure Lorentz transformations do not form a group as commutation of two boosts contains a rotation!!

But there is a Lorentz group!
How do we see it?

Say $A_i = \frac{1}{2}(J_i + iK_i)$, $B_i = \frac{1}{2}(J_i - iK_i)$

Then

$$\boxed{\begin{aligned} [A_i, A_j) &= i\epsilon_{ijk} A_k \\ [B_i, B_j) &= i\epsilon_{ijk} B_k \\ [A_i, B_j) &= 0 \end{aligned}}$$

Algebra of $SU(2) \times SU(2)$