Then, as before (see previous) we have

$$\boxed{\begin{aligned} e^{-iP_i x_i}\, a_p\, e^{iP_i x_i} &= a_p\, e^{ip_i x_i} \\ e^{-iP_i x_i}\, a_p^{+}\, e^{iP_i x_i} &= a_p^{+}\, e^{-ip_i x_i} \end{aligned}}$$

$\therefore$

$$e^{+iP_i x_i}\, \varphi(x_i)\, e^{-iP_i x_i} = \varphi(0)$$

So $\varphi(x_i) = e^{-iP_i x_i}\, \varphi(0)\, e^{iP_i x_i}$

$$\varphi(x) = e^{i(Ht - iP_i x_i)}\, \varphi(0)\, e^{-i(Ht - iP_i x_i)}$$

$$\Rightarrow \boxed{\varphi(x) = e^{iP\cdot x}\, \varphi(0)\, e^{-iP\cdot x}}$$

where $P^\mu = (H, P^i)$

Recall Heisenberg picture field:

$$\boxed{\varphi(x) = \varphi(x_i, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left[a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x} \right] \Big|_{\underline{\underline{p_0 = E_p}}}}$$

This expression makes clear "particle - wave" duality of field $\varphi(x)$.

$\varphi(x)$ → Hilbert space operator: creates/destroys particles, statistics commutation relations

$\varphi(x)$ → Linear combination of solutions of the Klein-Gordon <u>wave</u> equation

Have $e^{-ip_0 t}$ and $e^{ip_0 t}$ contributions

(w/ p_0 taking negative + positive values)

<u>NEGATIVE</u> + <u>POSITIVE</u> frequency modes

↙ coefficient is operator that <u>CREATES</u> particles in single-particle W.F.

↘ coefficient is operator that <u>DESTROYS</u> particles in single-particle W.F.

Note that both have positive energy

$$\underline{p_0 = E_p}$$

Now we have full single-particle wavefunction:

$$\boxed{\langle 0 | \underset{(14)}{\varphi(x)} | p_i \rangle = e^{ip\cdot x} = \psi(x)}$$

This may seem odd as normalization is relativistic.

Say ρ = probability of finding particle in a unit volume.

Then $$\int_{V=1} \rho \, d^3x = 1$$

So if $\underline{\rho = \psi^* \psi}$ and $\underline{\psi = A e^{ix_i p_i}}$

Then $|A| = 1$

$$\Rightarrow \boxed{\psi = e^{ip_i x_i}}$$

But not relativistic !!

So what does our relativistic wave-function represent ??

For L.I. normalization must have

$$S^\mu = (S^0, S^i)$$

time-component of 4-vector

<u>Need:</u>

$$\int S^0 \, dx^1 dx^2 dx^3 = 1$$

<u>or</u>

$$\frac{1}{4!} \int \epsilon_{\mu\nu\kappa\lambda} S^\mu dx^\nu dx^\kappa dx^\lambda = 1$$

$$\shortparallel$$

$$\int S^\mu n_\mu \, dV = 1$$

space-like hypersurface w/ normal n_μ.

Normalization clearly L.I. !!

Probability current:

$$S_\mu = \psi^* \overleftrightarrow{\partial_\mu} \psi = 2 p_\mu \psi^* \psi$$

$$\Rightarrow \int_V 2 p_0 \psi^* \psi \, d^3x = 1$$

So if $V = 1$ (1 particle in volume)

then $\psi = \frac{1}{\sqrt{2 p_0}} e^{i p_i x_i}$

if $V = 2 p_0$ (2p_0 particles in volume)

then $\psi = e^{i p_i x_i}$

$\therefore$ Normalization corresponds to $2 p_0$ particles per unit volume.

CAUSALITY

Now we are in a position to discuss

causality : signals should be bounded by c.

Consider:

$$\boxed{\langle 0|\,\varphi(x)\,\varphi(y)\,|0\rangle : \text{Amplitude for particle } \varphi \text{ to propagate from } y \text{ to } x. \equiv D(x-y)}$$

$$= \langle 0| \int \frac{d^3p}{(2\pi)^3} \frac{a_p e^{ix\cdot p}}{\sqrt{2E_p}} \int \frac{d^3p'}{(2\pi)^3} \frac{a^\dagger_{p'} e^{-iy\cdot p'}}{\sqrt{2E_{p'}}} |0\rangle$$

(other terms vanish)

$$\langle 0|\, a_p a^\dagger_{p'}\, |0\rangle = (2\pi)^3 \delta^{(3)}(p_i - p_i')$$

$$\Rightarrow \boxed{D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)}}$$

(We have shown that $D(x-y)$ is L.I. !!)

Now consider select values of $\underline{\underline{x-y}}$.

$\boxed{\underline{\text{time like case}}}$: $\underline{x^0 - y^0 = t}$ $\underline{x_i - y_i = 0}$

particle evolves only in time

(If evolution is time like, can always find this frame.)

$$D(x-y) = \frac{4\pi}{(2\pi)^3}\int_0^\infty \frac{dp\, p^2}{2\sqrt{p^2+m^2}}\, e^{-i\sqrt{p^2+m^2}\,t}$$

(change variables to E.) $\left(dp = d(\sqrt{E^2-m^2}) = \frac{dE}{\sqrt{E^2-m^2}}\right)$

$$= \frac{1}{2\pi^2}\int_m^\infty dE\,\frac{(E^2-m^2)\,E}{2E\sqrt{E^2-m^2}}\, e^{-iEt}$$

$$= \frac{1}{4\pi^2}\int_m^\infty dE\,\sqrt{E^2-m^2}\, e^{-iEt}$$

$$\underset{t\to\infty}{\sim} \underline{\underline{e^{-imt}}}$$

(Dominated by lower limit of integration as oscillates wildly!)

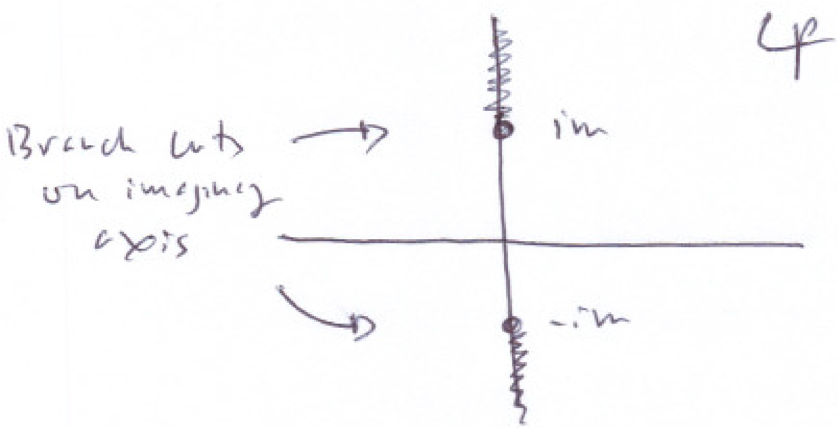
$\boxed{\text{Space Like Case}}$: $\underline{x^0 - y^0 = 0}$ $\quad$ $\underline{x_i - y_i = r_i}$

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{i p_i r_i}$$

$$= \frac{2\pi}{(2\pi)^3} \int_0^\infty \frac{dp\, p^2}{2E_p} \int_{-1}^{1} d(\cos\theta)\, e^{ipr\cos\theta}$$

$$= \frac{2\pi}{(2\pi)^3} \int_0^\infty \frac{dp\, p^2}{2E_p} \left(\frac{e^{ipr} - e^{-ipr}}{ipr} \right)$$

$$= \frac{-i}{2(2\pi)^2 r} \int_{-\infty}^{\infty} dp \frac{p e^{ipr}}{\sqrt{p^2+m^2}}$$



Define $\underline{\rho = -ip}$ $\quad$ $\underline{\rho_0 = -i(im) = m}$

$$= \frac{1}{4\pi^2 r^2} \int_m^\infty d\rho \frac{e^{-\rho r}}{\sqrt{\rho^2 - m^2}} \quad \underset{r\to\infty}{\sim} \quad e^{-mr}$$

Outside the light-cone propagation is exponentially vanishing but non-zero!!

<u>what's going on??</u>

We should be asking about <u>measurements</u> rather than particle propagation.

That is,

$$\boxed{\text{Can measurement at } A \text{ affect measurement at } B \text{ whose separation is spacelike?}}$$

Denoted by Q.M. operators:

$$[\hat{A}(x), \hat{B}(y)] \overset{??}{=} 0$$

If $= 0$ measurements cannot affect each other!

If $[\hat{A}(x), \hat{B}(y)] = 0$ for $\underline{(x-y)^2 < 0}$

then causality is preserved. (i.e. $(x_0-y_0)^2 < (x_i-y_i)^2$)

Consider $\hat{A} = \hat{B} = \varphi$

We know that $[\varphi(x_i), \varphi(y_i)] = 0$

for $x_0 = y_0$.

General case:

$$[\varphi(x), \varphi(y)] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{p'}}}$$

$$\times \left[\left(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x} \right), \left(a_{p'} e^{-ip'\cdot y} + a_{p'}^\dagger e^{ip'\cdot y} \right) \right]$$

$$= \int '' \int ''$$

$$\times \Big(\underbrace{[a_p^\dagger, a_{p'}]}_{-(2\pi)^3 \delta^{(3)}(p_i - p_i')} e^{i(p\cdot x - p'\cdot y)} + \underbrace{[a_p, a_{p'}^\dagger]}_{(2\pi)^3 \delta^{(3)}(p_i - p_i')} e^{-i(p\cdot x - p'\cdot y)} \Big)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right)$$

$$= D(x-y) - D(y-x)$$

$\left(\text{Note: } p_0 = E_p, \; p_0' = E_{p'} \right)$

Consider space-like separation $\underline{(x-y)^2 < 0}$.

We can Lorentz transform 2^{nd} term in
$[\varphi(x), \varphi(y)]$.

$$x_i - y_i \to -(x_i - y_i)$$

$$\Rightarrow \quad (x-y) \to -(x-y)$$

Then $[\varphi(x), \varphi(y)] = D(x-y) - D(x-y) = 0$

Causality is preserved!!

In light-like case $\underline{(x-y)^2 > 0}$ there is no continuous Lorentz transformation that takes

$$(x-y) \to -(x-y)$$

(we will show this later)

Amplitude doesn't vanish:

$$\sim e^{-imt} - e^{imt} \qquad (\text{for } x_i = y_i)$$

As all operators in the quantum field theory are functions of φ, we conclude that no measurement in K-G theory can affect another outside the light cone.

Aside: Complex K-G Field

Hermitian??

$$\boxed{\varphi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{-ip\cdot x} + b_p^+ e^{ip\cdot x} \right)}$$

b^+: creation operator for antiparticle (opposite charge, same mass)

b: destruction operator for antiparticle

$\therefore$ φ creates positively charged particles and destroys negatively charged ones.

(opposite for φ^+)

$$[\varphi(x), \varphi^+(y)] = D(x-y) - D(y-x)$$

propagation of negatively charged particle from y to x

propagation of positively charged particle from x to y

Causality requires that both types of propagation exist and be present. Each particle in QFT must have corresponding antiparticle w/ same mass and opposite quantum #s.

(Real K-G field is its own antiparticle!)

Back to: **Real K-G Propagator**

Since $[\varphi(x), \varphi(y)]$ is a c-number,

$$[\varphi(x), \varphi(y)] = \langle 0 | [\varphi(x), \varphi(y)] | 0 \rangle$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right)$$

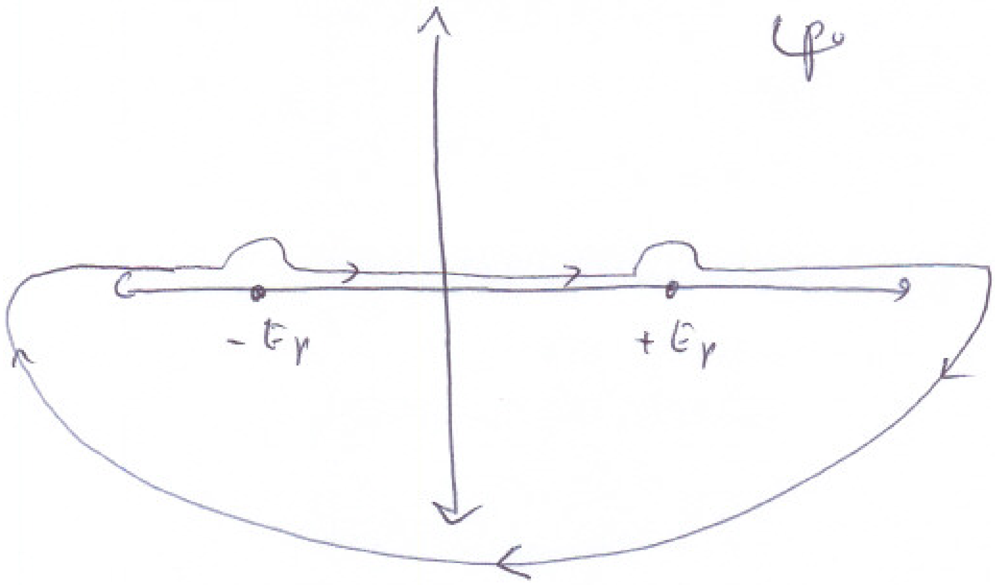
$$= \int \frac{d^3p}{(2\pi)^3} \left(\frac{1}{2E_p} e^{-ip\cdot(x-y)} \Big|_{p_0 = E_p} + \frac{1}{-2E_p} e^{-ip\cdot(x-y)} \Big|_{p_0 = -E_p} \right)$$

$$\underset{x^0 > y^0}{=} \int \frac{d^3p}{(2\pi)^3} \int \frac{dp^0}{2\pi i} \left(\frac{-1}{p^2 - m^2} \right) e^{-ip\cdot(x-y)}$$

CHECK

Integrand has poles at $\underline{p^2 - m^2 = 0}$

$$\Rightarrow \quad p_0^2 = E_p^2 \quad \Rightarrow \quad p_0 = \pm E_p$$



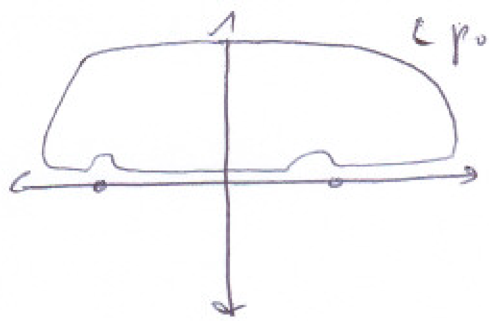
Pick up both poles !!

$$\frac{1}{p^2 - m^2} = \frac{1}{(p_0 + E_p)(p_0 - E_p)}$$

etc.

✓

Note: for $\underline{x_0 < y_0}$,



No poles !!

Hence, $[\phi(x), \phi(y)]$ vanishes for $x_0 < y_0$!!

(Retarded Green's fn.)

Say

$$\boxed{D_R(x-y) = \theta(x^0-y^0)\,\langle 0|[\varphi(x),\varphi(y)]|0\rangle}$$

$$\left(= \theta(x^0-y^0)\int\frac{d^3p}{(2\pi)^3}\int\frac{dp_0}{2\pi i}\left(-\frac{1}{p^2-m^2}\right)e^{-ip\cdot(x-y)}\right)$$

Consider:

$$(\Box+m^2)D_R(x-y) = \partial_\mu\Big(\partial^\mu(\theta(x^0-y^0))\langle 0|[\varphi,\varphi]|0\rangle$$

$$+\,\theta(x^0-y^0)\,\partial^\mu\langle 0|[\varphi,\varphi]|0\rangle\Big)$$

$$+\,m^2 D_R(x-y)$$

$$= \Box(\theta(x^0-y^0))\langle 0|[\varphi,\varphi]|0\rangle$$

$$+\,2\,\partial_\mu\theta(x^0-y^0)\,\partial^\mu\langle 0|[\varphi,\varphi]|0\rangle$$

$$+\,\theta(x^0-y^0)\,\Box\langle 0|[\varphi,\varphi]|0\rangle + m^2 D_R(x-y)$$

$$= \Box(\theta(x^0-y^0))\langle 0|[\varphi,\varphi]|0\rangle$$

$$+\,2\,\delta(x^0-y^0)\,\langle 0|[\pi(x),\varphi(y)]|0\rangle$$

$$+\,\theta(x^0-y^0)\,\cancelto{0}{(\Box+m^2)\langle 0|[\varphi(x),\varphi(y)]|0\rangle}$$

1st term is a bit tricky

equal-time commutator $\to 0$!

$$\left\{ \frac{\partial}{\partial x_0}\delta(x_0-y_0)[\varphi,\varphi] = \frac{\partial}{\partial x_0}\left[\delta(x_0-y_0)\cancelto{0}{[\varphi(x),\varphi(y)]}\right) - \delta(x_0-y_0)[\pi(x),\varphi(y)] \right\}$$

$$= -\delta(x^0-y^0)\langle 0|[\pi(x),\varphi(y)]|0\rangle + 2\delta(x^0-y^0)\langle 0|[\pi(x),\varphi(y)]|0\rangle$$

$$= \delta(x^0-y^0)\left(-i\,\delta^{(3)}(x_i-y_i)\right)$$

$$\Rightarrow \boxed{(\Box+m^2)D_R(x-y) = -i\,\delta^{(4)}(x-y)}$$