Let's switch to <u>Heisenberg picture</u>

so we can discuss role of <u>time</u> and therefore

address issue of <u>CAUSALITY</u>

$$\underbrace{\varphi(x)}_{(H)} = \varphi(x_i, t) = e^{iHt}\, \underbrace{\varphi(x_i)}_{(S)}\, e^{-iHt}$$

(same for $\pi(x) = \pi(x_i, t)$)

Heisenberg E.O.M. for operator $\mathcal{O}$:

$$\boxed{i \frac{\partial}{\partial t} \mathcal{O} = [\mathcal{O}, H]}$$

<u>CHECK</u>:

$$i \frac{\partial}{\partial t} \varphi(x_i, t) = [\varphi(x_i, t), H]$$

$$= [\varphi(x_i, t), \int d^3x' \left(\tfrac{1}{2}\pi^2(x_i', t) + \tfrac{1}{2}(\nabla_i' \varphi(x_i', t))^2 + \tfrac{1}{2} m^2 \varphi^2(x_i', t) \right)]$$

$$\left\{ \begin{array}{l} \text{Recall: } \quad [\varphi, \pi^2] = [\varphi, \pi]\pi + \pi[\varphi, \pi] \\ + \quad [\varphi(x_i, t), \pi(x_i', t)] = i\,\delta^{(3)}(x_i - x_i') \end{array} \right\}$$

$$\Rightarrow \quad i\frac{\partial \varphi(x_i, t)}{\partial t} = \int d^3x' \left(i\,\delta^{(3)}(x_i - x_i')\,\pi(x_i', t) \right)$$

$$(1) \qquad '' \qquad = i\,\pi(x_i, t)$$

Similarly,

$$(2) \qquad i\frac{\partial}{\partial t}\pi(x_i, t) = -i(-\nabla_i^2 + m^2)\,\varphi(x_i, t)$$

$$\frac{\partial (1)}{\partial t} \Rightarrow \quad i\frac{\partial^2}{\partial t^2}\varphi(x_i, t) = -i(-\nabla_i^2 + m^2)\,\varphi(x_i, t)$$

$$\Rightarrow \quad (\Box + m^2)\,\varphi(x_i, t) = 0$$

K-G equation

To find explicit expressions for $\varphi(x)$ and $\pi(x)$

Note: $\quad [H, a_p] = -\omega_p a_p$

$$\Rightarrow \quad H a_p = a_p H - \omega_p a_p$$

$$\Rightarrow \quad \underline{H a_p = a_p (H - E_p)}$$

Thus, $\quad H^2 a_p = H a_p (H - E_p) = a_p (H - E_p)^2$

etc.

$\Downarrow$

$$\boxed{\begin{aligned} H^n a_p &= a_p (H - E_p)^n \\ H^n a_p^+ &= a_p^+ (H + E_p)^n \end{aligned}}$$

$$\therefore \quad e^{iHt} a_p e^{-iHt} = a_p e^{i(H-E_p)t} e^{-iHt}$$

$$= \underline{a_p e^{-iE_p t}}$$

Similarly, $\quad e^{iHt} a_p^+ e^{-iHt} = \underline{a_p^+ e^{iE_p t}}$

Now we can get explicit expression for

$\varphi(x)$ from $\varphi(x_i)$

(H) (S)

$$\varphi(x) = e^{iHt}\,\varphi(x_i)\,e^{-iHt}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \Big[\underline{e^{iHt} a_p e^{-iHt}}\, e^{ip_i x_i} + \underline{e^{iHt} a_p^\dagger e^{-iHt}}\, e^{-ip_i x_i} \Big]$$

$$\boxed{\varphi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left[a_p e^{-ip\cdot x} + a_p^\dagger e^{+ip\cdot x} \right] \Big|_{p_0 = E_p}}$$

Note: $a_p = a_{p_i}$ are still defined as time-independent Schrödinger picture operators

Similarly, $\boxed{\pi(x) = \pi(x_i, t) = \frac{\partial}{\partial t}\varphi(x)}$

From Lorentz invariance might expect:

$$\varphi(x) = e^{ip\cdot x}\,\varphi(0)\,e^{-ip\cdot x}$$

How do we show this?

Recall: $$\underbrace{P_i = \int \frac{d^3p}{(2\pi)^3}\, p_i\, a_p^+ a_p}$$

Momentum Operator

$\Downarrow$

Eigenvalue is total momentum of the system.

momentum of single Fourier mode

For one-particle state of well-defined momentum,

$$P_i\, |p_i\rangle = p_i\, |p_i\rangle$$

$$[P_i, a_p^+] = \left[\int \frac{d^3p'}{(2\pi)^3}\, p_i'\, a_{p'}^+ a_{p'},\, a_p^+\right]$$

$$= \int \frac{d^3p'}{(2\pi)^3}\, p_i'\, a_{p'}^+ [a_{p'}, a_p^+]$$

$$= \int \frac{d^3p'}{(2\pi)^3}\, p_i'\, a_{p'}^+\, \delta^{(3)}(p_i - p_i') \Rightarrow$$

$$\boxed{\begin{aligned} [P_i, a_p^+] &= p_i\, a_p^+ \\ + \quad [P_i, a_p] &= -p_i\, a_p \end{aligned}}$$