K-G Field as Harmonic Oscillators

Now consider quantum field theory

Simplest case: real, free K-G field

As in N.R. Q.M., start with classical theory and quantize:

$$\boxed{\text{Reinterpret variables as operators that obey canonical commutation relations}}$$

$$\left\{ \begin{array}{lll} \text{Historically called } 2^{nd} \text{ quantization} & & \\ 1^{st} & \text{classical} \to \text{Q.M.} & (\psi) \\ 2^{nd} & \psi \to \text{field} & \end{array} \right\}$$

Here there is no wave function!

φ is classical field $\Rightarrow$ quantized once!!

After quantizing we can "solve" the theory by finding eigenvalues and eigenstates of $\mathcal{H}$.

Recall: $\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{m^2}{2}\varphi^2$

E.O.M. $(\Box + m^2)\varphi = 0$

$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla_i \varphi)^2 + \frac{1}{2}m^2\varphi^2$

Quantize!

$$\boxed{\varphi(x),\ \pi(x) \Rightarrow \text{OPERATORS}}$$

Commutation relations?

Recall system of N particles in Q.M.

Discrete system

$[q_i, p_j] = i\delta_{ij}$ $\qquad [q_i, q_j] = [p_i, p_j] = 0$

$i, j = 1 \dots N$

Continuous System??

$$\boxed{\begin{aligned} [\varphi(x_i), \pi(x_i)] &= i\,\delta^{(3)}(x_i - y_i) \\ [\varphi(x_i), \varphi(y_i)] &= [\pi(x_i), \pi(y_i)] = 0 \end{aligned}} \quad (*)$$

In Schrödinger "picture" φ, π ind. of time;

t-dependence carried by states

In Heisenberg picture (*) holds at equal time.

Now $\mathcal{H}$ is also an Operator.

SPECTRUM??

Expand K-G field in Fourier space:

$$\varphi(x_i, t) = \int \frac{d^3p}{(2\pi)^3}\, e^{i p_i x_i}\, \varphi(p_i, t)$$

(As φ is real, $\varphi^*(p_i) = \varphi(-p_i)$)

K-G equation:

$$(\Box + m^2)\,\varphi(x) = 0$$

$$\left(\frac{\partial^2}{\partial t^2} + p_i p_i + m^2\right)\varphi(p_i, t) = 0$$

E.O.M. for S.H.O. w/ frequency

$$\omega_p = \sqrt{p_i p_i + m^2}$$

Recall:

$$H_{S.H.O.} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2\varphi^2$$

$$\varphi = \frac{1}{\sqrt{2\omega}}(a + a^+) \qquad p = -i\sqrt{\frac{\omega}{2}}\,(a - a^+)$$

$$[a, a^+] = 1$$

Then, $H_{S.H.O.} = \omega\left(a^+ a + \frac{1}{2}\right)$

$$a|0\rangle = 0$$

$|0\rangle$ is eigenstate of $H_{S.H.O.}$ w/ eigenvalue $\frac{\omega}{2}$ (0-point energy)

How do we find other states?

Note: $[H_{S.H.O.}, a^+] = \omega a^+$

$[H_{S.H.O.}, a] = -\omega a$

Can show that $\underline{|n\rangle = (a^+)^n |0\rangle}$

$$\underline{H_{S.H.O.} |n\rangle = (n + \tfrac{1}{2}) \omega |n\rangle}$$

(Review this if rusty!!)
e.g. SAKURAI

<u>By analogy:</u>

$$\boxed{\varphi(x_i) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p e^{ip_i x_i} + a_p^+ e^{-ip_i x_i} \right)}$$

$(\omega_p \equiv \omega_{p_i})$

$$\boxed{\pi(x_i) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_p}{2}} \left(a_p e^{ip_i x_i} - a_p^+ e^{-ip_i x_i} \right)}$$

Each Fourier mode of field is an independent oscillator w/ its own a and a^+.

MASSAGE $\Rightarrow$

$$\phi(x_i) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p + a_{-p}^+ \right) e^{ip_i x_i}$$

$$\pi(x_i) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_p}{2}} \left(a_p - a_{-p}^+ \right) e^{ip_i x_i}$$

Again by analogy:

$$[a^+, a] = 1$$

$$\Downarrow$$

$$\boxed{[a_p, a_{p'}^+] = (2\pi)^3 \, \delta^{(3)}(p_i - p_i')}$$

Now Verify !!

$$[\varphi(x_i), \pi(x_i')] =$$

$$\int \frac{d^3p\, d^3p'}{(2\pi)^6} \left(-\frac{i}{2}\right) \sqrt{\frac{\omega_{p'}}{\omega_p}} \left([a_{-p}^+, a_{p'}] - [a_p, a_{-p'}^+] \right) e^{i(\vec{p}\cdot\vec{x} + \vec{p}'\cdot\vec{x}')}$$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^6} \left(-\frac{i}{2}\right) \sqrt{\frac{\omega_{p'}}{\omega_p}} \left(-(2\pi)^3 \delta^{(3)}(p+p') - (2\pi)^3 \delta^{(3)}(p+p') \right) \text{"}$$

$$= i \int \frac{d^3p\, d^3p'}{(2\pi)^3} \sqrt{\frac{\omega_{p'}}{\omega_p}}\, \delta^{(3)}(p+p')\, e^{i(\vec{p}\cdot\vec{x} + \vec{p}'\cdot\vec{x}')}$$

$$= i \int \frac{d^3p}{(2\pi)^3} e^{i p_i (x_i - x_i')}$$

$$= i\, \delta^{(3)}(x_i - x_i') \quad \boxed{\equiv}$$

Now express $\mathcal{H}$ in terms of ladder Operators

$$H = \int d^3x\, \mathcal{H} = \int d^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla_i \varphi)^2 + \frac{1}{2} m^2 \varphi^2 \right)$$

$$\left(\nabla_i \varphi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (i p_i) (a_p + a_{-p}^+) e^{i\vec{p}\cdot\vec{x}} \right)$$

$$H = \int d^3x \int \frac{d^3p\, d^3p'}{(2\pi)^6} e^{i(p_i+p_i')x_i} \Big\{ -\frac{1}{4}\sqrt{\omega_p\omega_{p'}}\,(a_p - a_{-p}^+)(a_{p'} - a_{-p'}^+)$$

$$- \frac{1}{4\sqrt{\omega_p\omega_{p'}}} (\vec{p}\cdot\vec{p}' - m^2)(a_p + a_{-p}^+)(a_{p'} + a_{-p'}^+) \Big\}$$

A.) $\Big($ Recall: $\underbrace{\int d^3x\, e^{i(\vec{p}+\vec{p}')\cdot\vec{x}} = (2\pi)^3 \delta^{(3)}(p_i + p_i')}$ $\Big)$

B.) (Do d^3p' integration)

$$= \int \frac{d^3p}{(2\pi)^3} \left[-\frac{\omega_p}{4} (a_p - a_{-p}^+) \times (a_{-p} - a_p^+) + \frac{(\vec{p}^2 + m^2)}{4\omega_p} (a_p + a_{-p}^+) \times (a_{-p} + a_p^+) \right]$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{\omega_p}{4} \Big[-(a_p a_{-p} + a_{-p}^+ a_p^+ - a_{-p}^+ a_{-p} - a_p a_p^+) + (a_p a_{-p} + a_{-p}^+ a_p^+ + a_p a_p^+ + a_{-p}^+ a_{-p}) \Big]$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{\omega_p}{4} \left(2 a_{-p}^+ a_{-p} + 2 a_p a_p^+ \right)$$

$(p \to -p)$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{\omega_p}{4} \left(a_p^+ a_p + a_p a_p^+ \right)$$

Finally,

$$\boxed{\underbrace{H = \int \frac{d^3p}{(2\pi)^3}\, \omega_p \left(a_p^+ a_p + \frac{1}{2} [a_p, a_p^+] \right)}}$$

$\sim \delta^{(3)}(0) \sim \infty$

c-number

$\sum_{\vec{p}} \frac{\omega_p}{2} \Rightarrow$ sum over all modes of the zero-point energy !!

Problematic in presence of gravity.

Commutators

$$[H, a_{p'}] = \int \frac{d^3p}{(2\pi)^3}\, \omega_p \, [a_p^+ a_p, a_{p'}]$$

$$\left\{ [a_p^+ a_p, a_{p'}] = [a_p^+, a_{p'}] a_p = -(2\pi)^3 \delta^{(3)}(p_i - p_i') a_p \right\}$$

$\Downarrow$

$$\boxed{[H, a_{p'}] = -\omega_{p'} a_{p'}} \qquad \boxed{[H, a_{p'}^+] = \omega_{p'} a_{p'}^+}$$

Now we can proceed as w/ S.H.O.

$a_p|0\rangle = 0$ for all p;

This is the vacuum w/ $E = 0$

(after dropping zero-point energy!)

$a_p^\dagger a_q^\dagger \cdots |0\rangle$ is eigenstate of H

w/ energy $\omega_p + \omega_q + \ldots$

How do we interpret eigenstates??

Recall: total momentum carried by K-G field:

$$P^i = -\int d^3x \, \pi(x_i) \, D_i \, \varphi(x_i)$$

$$= -\int d^3x \int \frac{d^3p \, d^3p'}{(2\pi)^6} e^{i(p_i + p_i')x_i} \frac{p_i'}{2} \sqrt{\frac{\omega_p}{\omega_{p'}}} (a_p - a_{-p}^\dagger)(a_{p'} + a_{-p'}^\dagger)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{p_i}{2} (a_p - a_{-p}^\dagger)(a_{-p} + a_p^\dagger)$$

$$\Rightarrow \quad \boxed{P^i = \int \frac{d^3p}{(2\pi)^3}\, p_i\, a_p^\dagger a_p}$$

So operator $a_p^\dagger$ creates state w/ momentum p_i and energy $\omega_p = \sqrt{p_i p_i + m^2}$ and operator $a_p^\dagger a_q^\dagger$ creates state w/ momentum $p_i + q_i$ and energy $\omega_p + \omega_q$, ETC.

It is natural to call these excitations <u>PARTICLES</u>

(not localized in space, rather in momentum as momentum eigenstates)

$$\omega_p = E_p = +\sqrt{p_i p_i + m^2}$$

POSITIVE!

<u>Note:</u> $a_p^\dagger a_q^\dagger |0\rangle = a_q^\dagger a_p^\dagger |0\rangle$

A single mode p can contain arbitrarily many particles. (Arbitrarily high H.O. levels)

$\Rightarrow$ K-G (scalar) fields obey

Bose-Einstein Statistics

Normalization

Vacuum: $\langle 0 | 0 \rangle = 1$

$|p_i\rangle \propto a_p^\dagger |0\rangle$

$\langle p_i | q_i \rangle = (2\pi)^3 \delta^{(3)}(p_i - q_i)$ (Weinberg uses this)

is not Lorentz invariant!

Recall Lorentz boosts!

E.g. Relative motion along z-direction

$x' = x, \; y' = y, \; z' = \dfrac{z + vt}{\sqrt{1 - v^2/c^2}}$ $\quad t' = \dfrac{t + vx/c^2}{\sqrt{1 - v^2/c^2}}$

In our "relativistic" notation $\Rightarrow$

$$x_3' = \gamma(x_3 + \beta x_0)$$
$$x_0' = \gamma(x_0 + \beta x_3)$$

$\gamma = \sqrt{1-\beta^2}$

$\beta = v/c$, $x_0 = ct$

Momentum boost along z ??

$$\left.\begin{matrix} z \\ t \end{matrix}\right\} \Rightarrow \begin{matrix} \text{conjugate} \\ \text{variables} \end{matrix} \left\{ \begin{matrix} p_3' = \gamma(p_3 + \beta E) \\ E' = \gamma(E + \beta p_3) \end{matrix} \right.$$

Aside: Identity

$$\boxed{\delta(f(x) - f(x_0)) \overset{?}{=}}$$

non-zero only for $x \to x_0$.

Taylor expand: $f(x) = f(x_0) + (x - x_0) f'(x_0) + \dots$

$$\delta(f(x) - f(x_0)) = \frac{1}{2\pi} \int dy\, e^{iy(f(x) - f(x_0))}$$

$$\| \qquad = \frac{1}{2\pi} \int dy\, e^{iy(x-x_0) f'(x_0)}$$

$$(y' = y |f'(x_0)|)$$

$$\boxed{\delta(f(x) - f(x_0)) = \frac{1}{|f'(x_0)|} \delta(x - x_0)}$$

Hence, under boost in 3-direction:

$$\delta^{(3)}(p_i - q_i) = \delta^{(3)}(p_i' - q_i') \cdot \frac{dp_3'}{dp_3}$$

$$= \delta^{(3)}(p_i' - q_i')\, \gamma \left(1 + \beta \frac{dE}{dp_3}\right)$$

$$\left\{ E = \sqrt{p_3^2 + m^2} \qquad \frac{dE}{dp_3} = \frac{1}{2}(p_3^2 + m^2)^{-\frac{1}{2}} (2 p_3) = \frac{p_3}{E} \right\}$$

$$= \delta^{(3)}(p_i' - q_i')\, \gamma (1 + \beta\, p_3/E)$$

$$= \delta^{(3)}(p_i' - q_i')\, \gamma\left(\frac{E + \beta p_3}{E}\right)$$

$$\Rightarrow \quad \delta^{(3)}(p_i - q_i) = \delta^{(3)}(p_i' - q_i')\, \frac{E'}{E}$$

{ Recall that $\delta^{(3)}$ is a volume and volumes are not invariant under boosts. }

But $\quad E_{p_i}\, \delta^{(3)}(q_i - p_i)$ is invariant !!

(i.e. covariant)

$\therefore$ define

$$|p_i\rangle = \sqrt{2E_p}\; a_p^\dagger |0\rangle$$

So $$\boxed{\langle p_i | q_i \rangle = 2E_p\, (2\pi)^3\, \delta^{(3)}(p_i - q_i)}$$

"COVARIANT" NORMALIZATION

(factor 2 convenient because of H.O. definition of a_p, $a_p^\dagger$)

With covariant normalization factors of $2E_p$ appear in places that may seem peculiar.

Covariant normalization implies:

$$\hat{U}(\Lambda)\,|p_i\rangle = |\Lambda p_i\rangle$$

($\hat{U}$ is unitary operator)

State vector transforms "covariantly"

$\hat{U}(\Lambda)$ can also act on operator rather than on state.

$$\hat{U}(\Lambda)\, a_p^\dagger\, \hat{U}^{-1}(\Lambda) = \sqrt{\frac{E_{\Lambda p}}{E_p}}\; a_{\Lambda p}^\dagger$$

Completeness ??

$$(\mathbb{1})_{1\text{-particle}} = \int \frac{d^3p}{(2\pi)^3}\, |p_i\rangle \frac{1}{2E_p} \langle p_i|$$

$$\langle q_i | q_i' \rangle = \int \frac{d^3p}{(2\pi)^3} \langle q_i | p_i \rangle \frac{1}{2E_p} \langle p_i | q_i' \rangle$$

$$= \int \frac{d^3p}{(2\pi)^3} 2E_p (2\pi)^3 \delta^3(q_i - p_i) \cancel{\frac{2E_p}{2E_p}}^{1} (2\pi)^3 \delta^3(p_i - q_i')$$

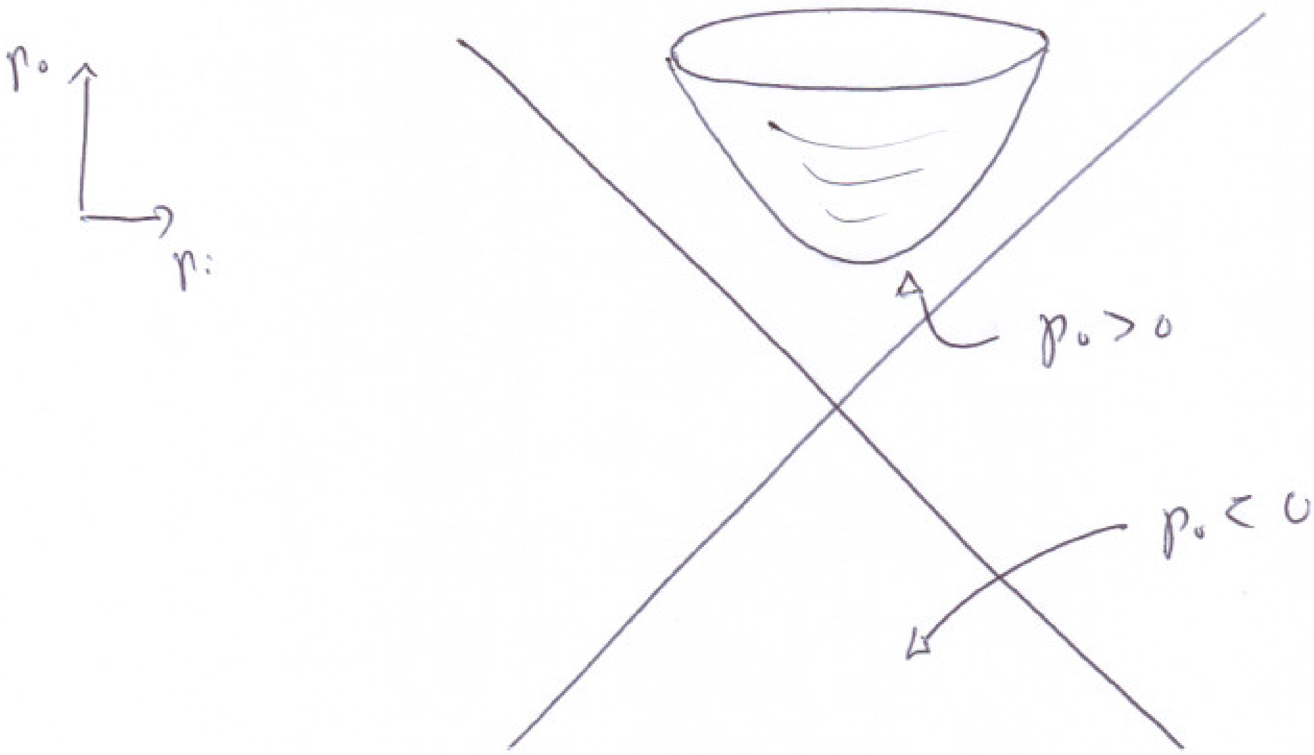
$$= 2E_q (2\pi)^3 \delta^3(q_i - q_i') \quad \boxtimes$$

Note:

$$\boxed{\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} = \int \frac{d^4p}{(2\pi)^4} (2\pi) \delta(p^2 - m^2) \Big|_{p^0 > 0}}$$

is Lorentz invariant 3-momentum integral in the sense that if $f(p)$ is L.I., then so is $\underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p}}\, f(p)$

Integral is over $p_0 > 0$ hyperboloid



$$p^2 = m^2 = p_0^2 - p_i p_i$$

$$\delta\left(p_0^2 - (p_i p_i + m^2)\right) \rightarrow \delta\left(\frac{p_0 - E_p}{2E_p}\right)$$

(AS BEFORE ...)

$$\therefore \int \frac{d^3p}{(2\pi)^3} \int dp_0 \, \delta(p^2 - m^2)_{p^0 > 0} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p}$$

Back to field $\varphi(x)$

$$\boxed{\varphi(x_i)\,|0\rangle = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_p}\,e^{-i p_i x_i}\,|p_i\rangle}$$

This state is a linear superposition of states of definite momentum p_i.

Other than factor $1/2E_p$, this is familiar non-relativistic expression for an eigenstate of position $|x_i\rangle$.

Interpretation:

> $\varphi(x_i)$ acting on $|0\rangle$ creates a particle at position x_i.

Consider

$$\langle 0|\,\varphi(x_i)\,|p_i\rangle = \langle 0| \int \frac{d^3p'}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{p'}}}\,(a_{p'} + a^{\dagger}_{-p'})\,e^{i\vec{p}'\cdot\vec{x}}$$
$$\times \sqrt{2E_p}\; a_p^{\dagger}\,|0\rangle$$

$$= \langle 0| \int \frac{d^3p'}{(2\pi)^3} \sqrt{\frac{E_p}{E_{p'}}} \, [a_{p'}, a_p^+] \, e^{i\vec{p}'\cdot\vec{x}} |0\rangle$$

$$= \langle 0| \int d^3p' \sqrt{\frac{E_p}{E_{p'}}} \, \delta^3(p'-p) \, e^{i\vec{p}'\cdot\vec{x}} |0\rangle$$

$$= e^{i\vec{p}\cdot\vec{x}}$$

$$\boxed{\langle 0| \varphi(x_i) | p_i\rangle = e^{i p_i x_i}}$$

Position space representation of <u>single-particle wavefunction</u> of the state $|p_i\rangle$.

(as in N.R. Q.M. $\langle x_i | p_i\rangle = e^{i p_i x_i}$ is wavefunction of state $|p_i\rangle$.)

S<u>ummar</u>y

Schrödinger picture:

$\underset{\sim}{\varphi(x_i)}$: Hermitian operator acting in a Hilbert space.

$$a_p^{+}|0\rangle = \frac{1}{\sqrt{2E_p}}\,|p_i\rangle$$

"momentum-space" particle state w/ momentum p_i and energy $E_p = \omega_p = \sqrt{p_i p_i + m^2}$

(not localized in space)

$\varphi(x_i)\,|0\rangle$ => superposition of single-particle states of definite momentum.

(particle at position x_i)