$\underline{d < 4}$ λ is relevant !!

However, consider $\Delta\lambda$ in d dimensions:

$$\boxed{\lambda' = \left[\lambda - \frac{3\lambda^2}{(4\pi)^{d/2}\,\Gamma(\frac{d}{2})} \left(\frac{b^{d-4} - 1}{4-d} \right) \Lambda^{d-4} \right] b^{d-4}}$$

2 competing effects !

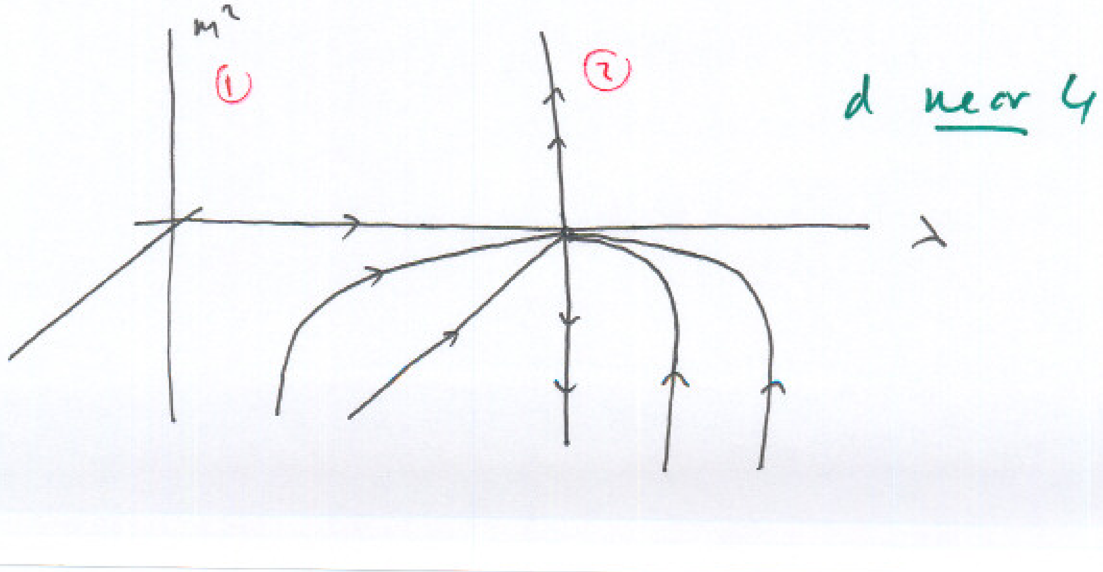
decrease due to higher order effects

increase due to rescaling

$\therefore$ There is value of λ where there is precise balance between effects: λ unchanged as we integrate out degrees of freedom.

Corresponding λ is another 2nd fixed point of the R.G.

As $d \to 4$ this fixed point merges with the first one.



Comments

- There are also strongly coupled fixed points

In nature, all QFT's are controlled by free-field fixed points.

(This is mysterious)

- As $m^2\varphi^2$ term is relevant, coeff[icien]t changes rapidly under RG flow. In order to have natural value of m^2 in low-energy theory we have to imagine that value of m^2 in original $\mathcal{L}$ is adjusted delicately (fine tuned).

"Hierarchy problem" in S.M.

Effective field theory and Matching

We have shown that starting out with $\mathcal{L}(\varphi)$ we can successively "integrate out" high-momentum modes and end up with $\mathcal{L}_{eff}(\varphi)$ which contains an infinite number of operators.

Consider instead a theory of several fields, one of which is much heavier than the rest.

Consider

$$\boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{1}{2}(\partial_\mu \Phi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{1}{2}M^2\Phi^2 - \frac{1}{2}\kappa\,\varphi^2\Phi}$$

(recall HW 4)

Say $\mathcal{L}$ is the "fundamental theory".

Consider $\varphi\varphi \to \varphi\varphi$ at $E \ll M$

We can work with $\mathcal{L}$ directly or we can work with $\mathcal{L}_{eff}$ with Φ "integrated out."

To ensure that predictions of the fundamental theory and the effective theory agree we have to

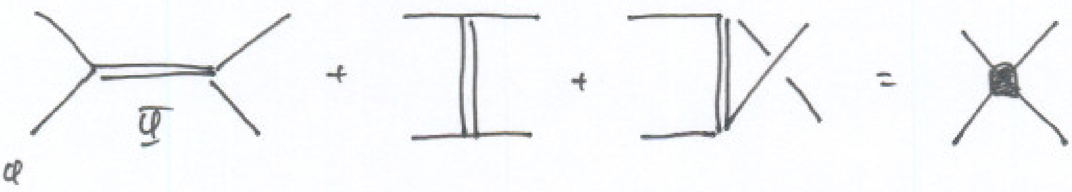
Match correlation functions in the 2 theories.

formally,

$$\boxed{\int D\varphi \int D\Phi \; e^{iS[\varphi,\Phi]} = \int D\varphi \; e^{iS_{eff}[\varphi]}}$$

(cw Wilsonian renormalization.)

Consider tree-level matching:



full theory E.F.T

It is clear that at tree level Φ exchange generates a φ^4 interaction in the effective theory:

$$\boxed{\mathcal{L}^0_{eff} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m^2\varphi^2 - c_0\left(\frac{k}{M}\right)^2\frac{\varphi^4}{4!} + \dots}$$

c_0 dimensionless $\sim \mathcal{O}(1)$

$$\dots \Rightarrow \frac{k^2}{M^4}\varphi^2\partial^2\varphi^2$$

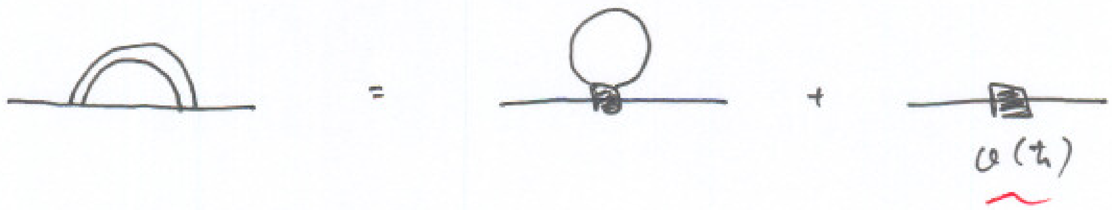
{ e.g.  $\sim \frac{k^2}{p^2-M^2} \sim \frac{k^2}{M^2} + \frac{k^2p^2}{M^4} + \mathcal{O}(p^4)$ }

For non-relativistic theory:

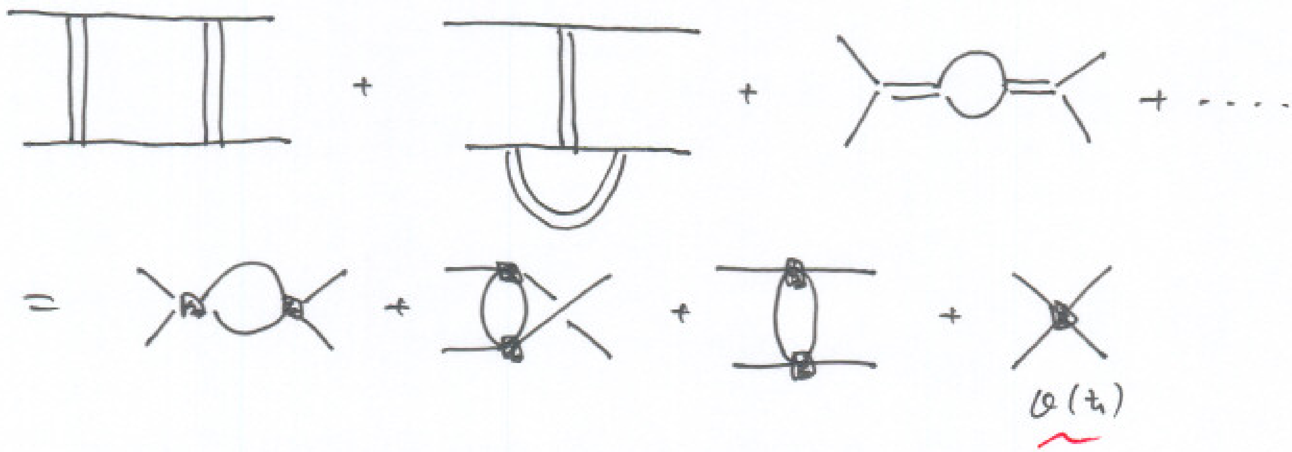
$$\frac{e^{-Mr}}{r} \longrightarrow c_0\,\delta^3(r) + c_2\nabla^2\delta^3(r) + \dots$$

Now consider match at 1-loop ($\mathcal{O}(\hbar)$).

2-pt.



4-pt.



Match using dim reg (w $\overline{MS}$) at $\underline{\mu = M}$.

↳ (subtracts $\frac{1}{\epsilon}$ + finite junk)

At 1-loop:

$$\boxed{\begin{aligned}\mathcal{L}_{eff} &= \frac{1}{2}\left(1 + a_1 \frac{\kappa^2}{16\pi^2 M^2}\right)(\partial_\mu \varphi)^2 - \frac{1}{2}\left(m^2 + b_1 \frac{\kappa^2}{16\pi^2}\right)\varphi^2 \\ &\quad - \left[c_0\left(\frac{\kappa^2}{M^2}\right) + c_1\left(\frac{\kappa^4}{16\pi^2 M^4}\right)\right]\frac{\varphi^4}{4!} + \dots\end{aligned}}$$

{ We can rescale φ to put kinetic term in canonical form }

Now we can use $\mathcal{L}_{eff}$ to compute $\varphi\varphi \to \varphi\varphi$ at $E \ll M$.

Comments

- $\boxed{\text{loop expansion} = \text{expansion in } \frac{\kappa^2}{16\pi^2 M^2}}$

  If this is small, perturbation theory is valid.

- We didn't write down the "irrelevant" operators. These effects, $\mathcal{O}\left(\frac{E^2}{M^2}\right)$, may be as important at low energy as subleading corrections to φ^4 operator.

- Note:

$$\boxed{m^{2\prime} = m^2 + b_1 \frac{\kappa^2}{16\pi^2}}$$

↑ not $\propto m^2$!!

Unnatural to expect $m^{2\prime} \ll \frac{\kappa^2}{16\pi^2}$

(this would require fine-tuning the 2 contributions)

[For $\kappa + m$ small φ is naturally light as $\mathcal{L}$ has symmetry $\varphi \to \varphi + \text{constant}$]

- Coefficients in $\mathcal{L}_{eff}$ are regularization scheme dependent. However physics (e.g. $\sigma(\varphi\varphi \to \varphi\varphi)$) is not!!

- Graphs in both theories have pieces that are non-analytic in m and p; e.g. $\log m$, $\log p$. These must cancel in matching so that $\mathcal{L}_{eff}$ has local expansion in $\frac{1}{M}$.

  (or rather these non-analytic pieces are universal)

Utility of E.F.T.

{ Say we don't know the fundamental theory with an explicit Φ, but we observe interacting φ's at low energies: }

Write down $\mathcal{L}_{eff}$ and determine coefficients experimentally. In this way one can infer existence of Φ.

Analogy.

φ represents observed SM particles

Φ represents the Higgs Boson

Example: Euler - Heisenberg E.F.T.

Consider Q.E.D :

$$\boxed{\mathcal{L}_{QED}(A_\mu, \psi) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \bar{\psi}(i\not{D} + m_e)\psi}$$

Many practical applications of E & M involve interactions of photons with macroscopic electric charge and current distributions at $E, p \ll m_e$

If we "integrate out" electron, we are left with an E.F.T of interacting photons.

$\mathcal{L}_{eff}(A_\mu)$ is local, hermitian and invariant under Lorentz, gauge ($U(1)$), C, P transformations:

$$\boxed{\begin{aligned}\mathcal{L}_{eff} = & -\frac{Z}{4} F_{\mu\nu} F^{\mu\nu} \\ & + \frac{a}{m_e^2} F_{\mu\nu} \Box F^{\mu\nu} + \frac{a'}{m_e^2} \partial_\mu F^{\mu\nu} \partial^\lambda F_{\lambda\nu} \\ & + \frac{b}{m_e^4} (F_{\mu\nu} F^{\mu\nu})^2 + \frac{c}{m_e^4} (F_{\mu\nu} \tilde{F}^{\mu\nu})^2 + \cdots\end{aligned}}$$

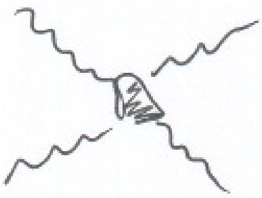
(recall $\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\lambda\rho} F^{\lambda\rho}$)

(Z, a, a', b, c, etc. dimensionless)

$$\left\{ \begin{array}{c} \text{Notice no odd number of } F\text{'s :} \\ C: \; F \to -F \end{array} \right\}$$

Light-by-Light Scattering

Interactions at $\mathcal{O}\left(\frac{1}{m_e^4}\right)$ contribute to $\gamma\gamma \to \gamma\gamma$ scattering.



$\sim b, c$

As A_μ only appears through $F_{\mu\nu}$:

each $A_\mu \to 1$ derivative $= 1$ power of momentum.

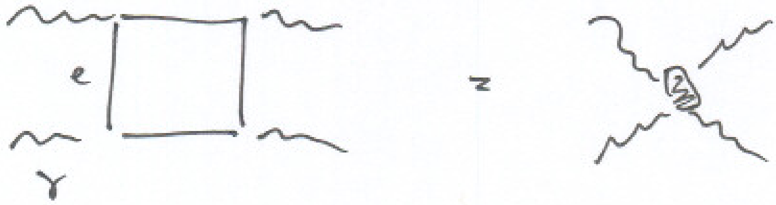
$\boxed{\text{EFT can be formulated as expansion in powers of } p}$

In C.M.S. one finds:

$$\boxed{\frac{d\sigma_{\gamma\gamma}}{d\Omega} = \frac{278}{65\pi^2}\left[(b+c)^2 - (b-c)^2\right]\left(\frac{E_{cm}^6}{m_e^8}\right)(3+\cos^2\theta)^2 \times \left(1 + \mathcal{O}\left(\frac{E_{cm}^4}{m_e^4}\right)\right)}$$

This holds even if Q.E.D is strongly coupled !!

One can obtain b, c by matching to full theory:



$$\Rightarrow \quad \boxed{b = \frac{4}{7}c = \frac{\alpha^2}{90}}$$

$\Rightarrow$ standard result

$$\boxed{\frac{d\sigma_{\gamma\gamma}}{d\Omega} = \frac{139}{4\pi^2}\left(\frac{\alpha^2}{90}\right)^2\left(\frac{E_{cm}^6}{m_e^8}\right)(3+\cos^2\theta)^2\left[1+\mathcal{O}\left(\frac{E_{cm}^4}{m_e^4}\right)\right]}$$

Effective field theories are designed to reproduce all of the infrared (light particle) physics of the full theory, while distorting the high-energy (ultraviolet) behavior to make calculations simpler!!

Beautiful universal technology!