Renormalization à la Wilson

We have seen that in Q.F.T., quantum fluctuations at arbitrarily short distances appear through the effect of virtual quanta with arbitrarily high momenta in Feynman diagrams.

Renormalizability $\Rightarrow$ loop integrals over virtual momenta are always dominated by momenta comparable to the "characteristic" external particle momenta.

Let's investigate why this is so in a more physically intuitive picture due to Ken Wilson.

Path Integral formulation using sharp cutoffs on momenta

Consider $\lambda\varphi^4$ theory.

$$Z[J] = \int D\varphi \, e^{i\int(\mathcal{L} + \varphi J)} = \left(\prod_k \int d\varphi(k)\right) e^{i\int(\mathcal{L} + \varphi J)}$$

Basic integration variables are Fourier modes $\varphi(k)$.

$$\varphi(k) = \int d^4x \, e^{ik\cdot x} \, \varphi(x)$$

A sharp cutoff restricts number of integration variables in path integral.

With Λ,

integrate over $\varphi(k)$ with $|k| \le \Lambda$; i.e.

$$\boxed{\varphi(k) = \begin{cases} \varphi(k) & |k| \le \Lambda \\ 0 & |k| > \Lambda \end{cases}}$$

Let's define Euclidean form of path integral:

$$\int d^4x \left(\mathcal{L} + J\varphi \right) = \int d^4x \left(\tfrac{1}{2}(\partial_\mu \varphi)^2 - \tfrac{1}{2} m^2 \varphi^2 - \frac{\lambda}{4!} \varphi^4 + J\varphi \right)$$

Wick Rotation: $x^2 = t^2 - |x_i|^2$

$$\Downarrow \quad \text{W.R.}$$

$$-(x^0)^2 - |x_i|^2 = -|x_E|^2$$

$\{$ Recall: $l^0 = i l^0_E \quad l_i = l_{iE}$ $\}$

$\therefore$ Euclidean form is:

$$i\int d^4x_E\,(\mathcal{L}_E - J\varphi) = i\int d^4x_E\left[\tfrac{1}{2}(\partial_{E\mu}\varphi)^2 + \tfrac{1}{2}m^2\varphi^2 + \tfrac{\lambda}{4!}\varphi^4 - J\varphi\right]$$

and

$$\boxed{Z[J] = \int D\varphi\,\exp\left[-\int d^4x_E\,(\mathcal{L}_E - J\varphi)\right]}$$

We would like to carry out integration over the high-momentum modes of φ:

$$Z[0] = \int [D\varphi]_\Lambda \exp\left[-\int d^dx\left[\tfrac{1}{2}(\partial_\mu\varphi)^2 + \tfrac{1}{2}m^2\varphi^2 + \tfrac{\lambda}{4!}\varphi^4\right]\right]$$

$$[D\varphi]_\Lambda = \prod_{|k|<\Lambda} d\varphi(k)$$

{ Euclidean path integral for $\lambda\varphi^4$ in d-dimensions }

m and λ are "bare parameters"

$\Downarrow$

no counterterms "a priori"

Let's divide up the integration regions into

2 groups

We want to integrate over variables $\hat{\varphi}(k)$

where:

$$\boxed{\hat{\varphi}(k) = \begin{cases} \varphi(k) & \text{for } b\Lambda \leq |k| < \Lambda \\ 0 & \text{otherwise} \end{cases}}$$

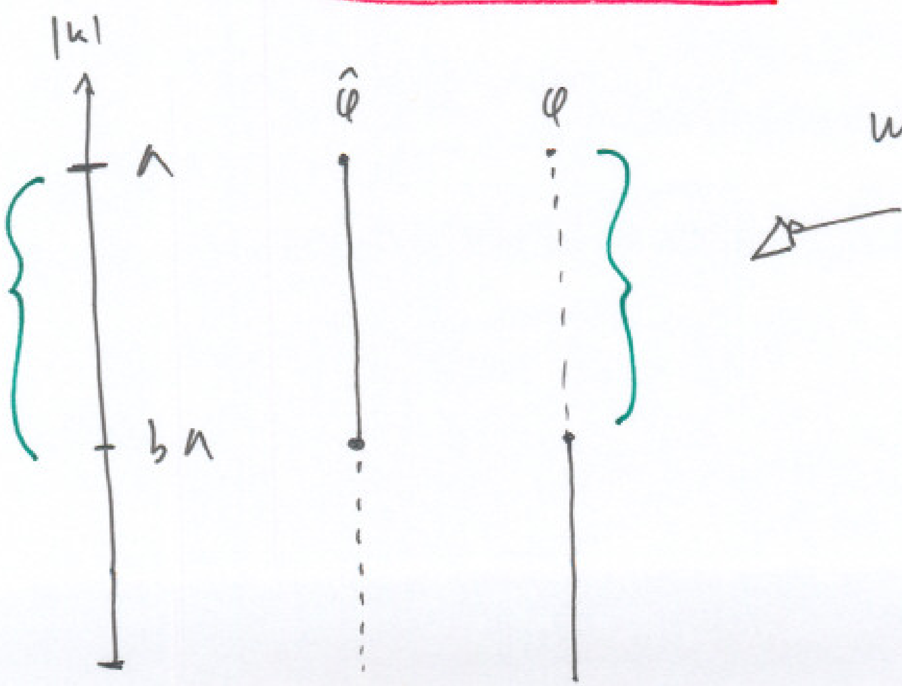
b : fraction < 1

Define new $\varphi(k)$:

$$\boxed{\varphi(k) = \begin{cases} \varphi(k) & \text{for } |k| < b\Lambda \\ 0 & \text{for } |k| > b\Lambda \end{cases}}$$

We can achieve this separation by taking

$$\boxed{\varphi \longrightarrow \varphi + \hat{\varphi}}$$



Want to "integrate out" these modes

So with $\varphi \to \varphi + \hat{\varphi}$ we have

$$Z = \int D\varphi \int D\hat{\varphi} \exp\left\{ - \int d^d x \left[\frac{1}{2}(\partial_\mu \varphi + \partial_\mu \hat{\varphi})^2 + \frac{1}{2} m^2 (\varphi + \hat{\varphi})^2 + \frac{\lambda}{4!}(\varphi + \hat{\varphi})^4 \right] \right\}$$

$\Downarrow$

$$\boxed{Z = \int D\varphi \, e^{-\int \mathcal{L}(\varphi)} \int D\hat{\varphi} \exp\left\{ - \int d^d x \left[\frac{1}{2}(\partial_\mu \hat{\varphi})^2 + \frac{1}{2} m^2 \hat{\varphi}^2 + \lambda \left(\frac{1}{6}\varphi^3 \hat{\varphi} + \frac{1}{4}\varphi^2 \hat{\varphi}^2 + \frac{1}{6}\varphi \hat{\varphi}^3 + \frac{1}{4!}\hat{\varphi}^4 \right) \right] \right\}}$$

{ Note: $\varphi \hat{\varphi}$ terms vanish : orthogonality }

If we do integration over $\hat{\varphi}$ we will be left with:

$$\boxed{Z = \int [D\varphi]_{b\Lambda} \exp\left\{ - \int d^d x \, \mathcal{L}_{eff} \right\}}$$

{ involves only components of φ w/ $|k| < b\Lambda$ }

We expect, in perturbation theory,

$$\underbrace{\mathcal{L}_{eff}(\varphi) = \mathcal{L}(\varphi) + \mathcal{O}(\lambda)}$$

compensate for removal of high-energy modes!!

Eg $\quad \mathcal{L}_0 = \frac{1}{2}(\partial_\mu \hat{\varphi})^2 \qquad$ Free

$\mathcal{L}' = \frac{1}{2} m^2 \hat{\varphi}^2 + \mathcal{O}(\lambda) \qquad$ perturbation

$\qquad\qquad\qquad\qquad$ (as $m^2 \ll a^2$)

Think of kinetic term in momentum space:

$\mathcal{L}_0 = -\frac{1}{2} \hat{\varphi} \Box \hat{\varphi} \qquad$ when $\hat{\varphi}(x) = \int \frac{d^d k}{(2\pi)^d} e^{-ik\cdot x} \hat{\varphi}(k)$

$$\Rightarrow \boxed{\int d^d x \, \mathcal{L} = -\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \hat{\varphi}(k) \, k^2 \, \hat{\varphi}(k)}$$

$\Downarrow$ propagator

$$\langle \hat{\varphi}(k) \hat{\varphi}(p) \rangle = \frac{\int D\hat{\varphi} \, \hat{\varphi}(k) \hat{\varphi}(p) \, e^{-\int \mathcal{L}_0}}{\int D\hat{\varphi} \, e^{-\int \mathcal{L}_0}} \Leftarrow \text{Have expanded } \mathcal{L}' \text{ !!}$$

$$= \frac{1}{k^2} (2\pi)^d \delta^{(d)}(k+p) \quad \Theta(k)$$

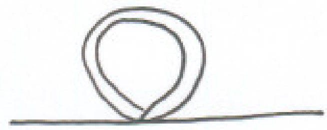
$$\left\{ \Theta(k) = \begin{cases} 1 & b\Lambda \le |k| < \Lambda \\ 0 & \text{otherwise} \end{cases} \right\}$$

{ This is just the momentum-space propagator for a massless particle. }

We can assign Feynman rule to these propagators:

 $\sim \dfrac{1}{k^2}\,\Theta(k)$

We then have:

$(*)$ $$-\int d^dx \, \underbrace{\frac{\lambda}{4!}\,\varphi^2\,\hat{\varphi}\hat{\varphi}} =$$ 

$$= -\frac{1}{2}\int d^dx \, \mu\,\varphi^2 \qquad \left\{= -\frac{1}{2}\int \frac{d^d\bar{k}}{(2\pi)^d}\,\mu\,\varphi(\bar{k})\varphi(-\bar{k})\right\}$$

in momentum space

where $$\mu = \frac{\lambda}{2}\int_{b\Lambda \le |k| \le \Lambda} \frac{d^dk}{(2\pi)^d}\,\frac{1}{k^2}$$

$$\left\{ \text{with a sharp cutoff,} \quad \mu = \frac{\lambda}{(4\pi)^{d/2}\,\Gamma\left(\frac{d}{2}\right)}\,\frac{\left(1-b^{d-2}\right)}{d-2}\,\Lambda^{d-2} \right\}$$ $\Leftarrow$ quadratic divergence in $d=4$

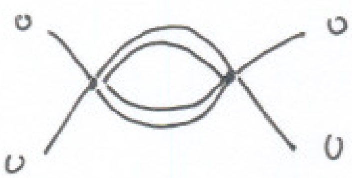
Hence, this particular contraction $(*)$ generates a correction to the mass term.

Indeed, we can "re-exponentiate" this term:

$$\Rightarrow \quad \exp\left(- \int d^d x \, \tfrac{1}{2} \mu \varphi^2 + \dots\right) \qquad (*)$$

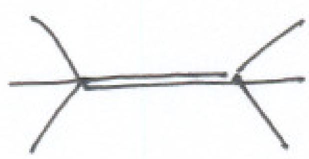
{other terms in this expansion also appear!!}

At $\mathcal{O}(\lambda^2)$ we can write down diagrams:



$$= -\frac{1}{4!} \int d^d x \, \xi \, \varphi^4$$

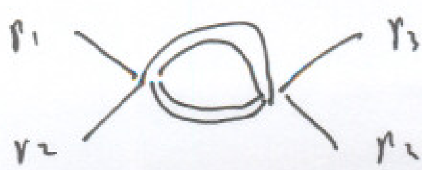
$$\left\{ \xi = -\frac{3\lambda^2}{16\pi^2} \lg \frac{1}{b} \right\}$$



$$= \int d^d x \, \chi \, \varphi^6 \qquad \{\chi \sim \lambda^2\}$$

{Can also have ()^2 which gives additional terms in $(*)$}

We also generate derivative interactions:



$$= -\frac{1}{4} \int d^d x \, \eta \, \varphi^2 (\partial_\mu \varphi)^2$$

In general, procedure of "integrating out" $\hat{\varphi}$ generates all possible interaction of the fields and their derivatives

Diagrammatic expansion is simplified by resumming as an exponential; schematically:

$$\mathcal{L}_{eff} = \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{1}{2} m^2 \varphi^2 + \frac{1}{4!} \lambda \varphi^4 + \left\{ \begin{matrix} \text{sum of} \\ \text{connected} \\ \text{diagrams} \end{matrix} \right\}$$

In terms of operators:

$$\boxed{\int d^d x \, \mathcal{L}_{eff} = \int d^d x \left[\frac{1}{2}(1+\Delta Z)(\partial_\mu \varphi)^2 + \frac{1}{2}(m^2 + \Delta m^2)\varphi^2 + \frac{1}{4!}(\lambda + \Delta\lambda)\varphi^4 + \Delta C (\partial_\mu \varphi)^4 + \Delta D \varphi^6 + \ldots \right]}$$

We see that when we integrate out $\hat{\varphi}$ we generate all possible non-renormalizable interactions, even though we started with only renormalizable interactions!!

Are these additional contributions under control??

Let's consider the renormalization group flow of the various couplings.

Rescale: $k' = \frac{k}{b}$ $\qquad x' = x b$

(hence k' is integrated over $|k'| < \Lambda$)

Then

$$\boxed{\int d^dx\, \mathcal{L}_{eff} = \int d^dx'\, b^{-d} \Big[\tfrac{1}{2}(1+\Delta Z)\, b^2 (\partial'_\mu \varphi)^2 + \tfrac{1}{2}(m^2+\Delta m^2)\varphi^2 + \tfrac{1}{4!}(\lambda+\Delta\lambda)\varphi^4 + \Delta C\, b^4 (\partial'_\mu\varphi)^4 + \Delta D\, \varphi^6 + \dots \Big]}$$

Notice that all terms beyond the first are perturbations!

Rescale kinetic term:

$$\boxed{\varphi' = \left[b^{2-d}(1+\Delta Z) \right]^{\frac{1}{2}} \varphi}$$

We then have:

$$\boxed{\int d^dx\, \mathcal{L}_{eff} = \int d^dx' \Big[\tfrac{1}{2}(\partial'_\mu \varphi')^2 + \tfrac{1}{2} m'^2 \varphi'^2 + \tfrac{1}{4!}\lambda' \varphi'^4 + C'(\partial'_\mu \varphi')^4 + D' \varphi'^6 + \dots \Big]}$$

$$m'^2 = (m^2 + \Delta m^2)(1+\Delta Z)^{-1} b^{-2}$$

$$\lambda' = (\lambda + \Delta\lambda)(1+\Delta Z)^{-2} b^{d-4}$$

$$C' = (C + \Delta C)(1+\Delta Z)^{-2} b^{d}$$

$$D' = (D + \Delta D)(1+\Delta Z)^{-3} b^{2d-6}$$

$$\vdots$$

Now $\mathcal{L}_{\text{eff}}$ is same as original $\mathcal{L}$ but with $C, D \neq 0$. But these additional pieces are small if perturbation theory in λ is valid.

$$\boxed{\text{integrating out high momentum modes + rescaling} = \text{transformation of } \mathcal{L}}$$

We could now integrate over another "shell" of momentum space. If we take shells infinitesimally thin, $b \to 1$, then transformations become continuous

$\Downarrow$

$$\boxed{\begin{array}{c}\text{integrating out high momentum modes} \\ \Downarrow \\ \text{trajectory (or flow) in space of all possible } \mathcal{L}\text{'s}\end{array}}$$

Renormalization group flow

{not group as non-invertible}

Notice that we can use original $\mathcal{L}$ or $\mathcal{L}_{\text{eff}}$ to compute correlators at $p \ll \Lambda$.

Loop effects in $\mathcal{L}$ = tree diagrams in $\mathcal{L}_{\text{eff}}$

Consider an RG flow:

Consider $\mathcal{L}$ near point where $m^2 = \lambda = C = D = \ldots = 0$

$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \varphi)^2$ is a fixed point of the renormalization group.

Near $\mathcal{L}_0$ keep only terms that are linear in perturbations:

$$\boxed{m'^2 = m^2 b^{-2} \qquad \lambda' = \lambda b^{d-4} \qquad C' = C b^{d} \qquad D' = D b^{2d-6} \quad \ldots}$$

As $b < 1$, $b^{\text{negative}} \to$ grows RELEVANT

$b^{\text{positive}} \to$ decay IRRELEVANT

If $\mathcal{L}$ has growing coefficients, $\mathcal{L}$ will move away from $\mathcal{L}_0$.

Clearly φ^2 operator is always RELEVANT.

φ^4 is relevant if $d < 4$

φ^4 is MARGINAL if $d = 4$

?

higher-order corrections determine whether it grows or decays.

Generally,

$$\boxed{C'_{N,M} = b^{N(d/2-1)+M-d}\, C_{N,M}}$$

N: powers of φ
M: # of derivatives

e.g. $\quad C' = C\, b^d \quad$ irrelevant

$D' = D\, b^{2d-6} \quad \left\{ \begin{array}{ll} \text{irrelevant} & d > 3 \\ \text{marginal} & d = 3 \\ \text{relevant} & d < 3 \end{array} \right\}$

Note that
Relevant = super renormalizable
marginal = renormalizable

Notice different perspective between Wilson RG and our discussion of renormalization of $\lambda\varphi^4$ with dimensional regularization.

Wilson: QFT is defined fundamentally with a cutoff Λ that has physical significance.

Previously we wanted to get rid of Λ as soon as possible, and we claimed that non-renormalizable interactions would spoil this!

The Wilson viewpoint we have shown that in vicinity of fixed point, arbitrary complicated $\mathcal{L}$ at scale Λ reduces to $\mathcal{L}$ with only renormalizable interactions

$\Uparrow$

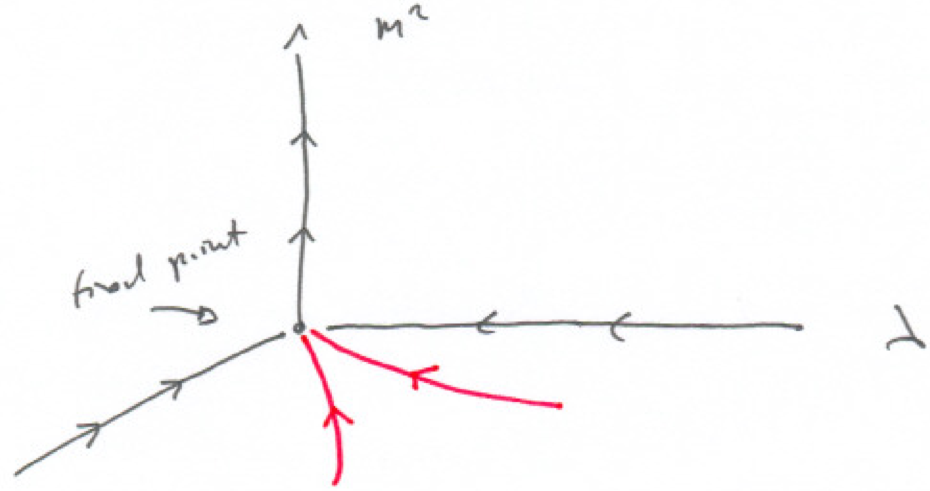
This explains renormalizability!!

Of course, entire analysis is tied to perturbation theory in small coupling: Strong coupling can alter this analysis.

Consider: RG flow near $\mathcal{L}_0$ fixed point in $\lambda\varphi^4$.

3 cases.

$d > 4$ only relevant operator is φ^2.



mass term increases in importance

$d=4$

Here φ^4 is marginal. Need to look at $\Delta\lambda$.

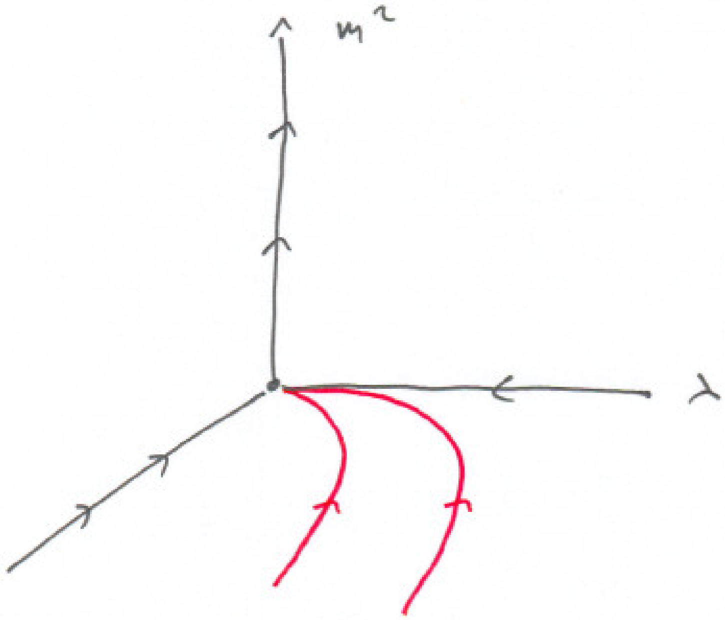
$$\lambda' = (\lambda + \Delta\lambda)(1 + \underbrace{\Delta z}_{O(\lambda^2)})^{-2}\, b^{d-4}$$

$$\Delta\lambda = -\frac{3\lambda^2}{16\pi^2}\log\frac{1}{b}$$

$$\Rightarrow \quad \boxed{\lambda' = \lambda - \frac{3\lambda^2}{16\pi^2}\log\frac{1}{b}}$$

$\therefore$ λ slowly decreases as we integrate out high-momentum modes $(b<1)$

(Notice that in Wilson's approach this diagram  is well defined)



1 slowly decaying direction