(2/7/07)

Classical Field Theory

Lagrangian Mechanics

Action $\qquad S = \int dt\, L(q_i(t), \dot{q}_i(t))$

$$\frac{\delta S}{\delta q_i(t)} = 0 \quad \Rightarrow \quad \text{E.O.M.}$$

Explicitly,

$$\delta S = \int dt \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right)$$

$$= \int dt \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i + \partial\text{-terms} = 0$$

E.O.M.:

$$\boxed{\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0}$$

Euler-Lagrange Eqns.

EXAMPLE: S.H.O. 1-d

$$L = \frac{1}{2}\dot{x}^2 - \frac{\omega^2}{2}x^2$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = -\omega^2 x - \ddot{x} = 0$$

$$\Rightarrow \ddot{x} = -\omega^2 x$$

Lagrangian Field Theory

Scalar field $\varphi = \varphi(x^\mu)$ = continuous family of D.O.F.

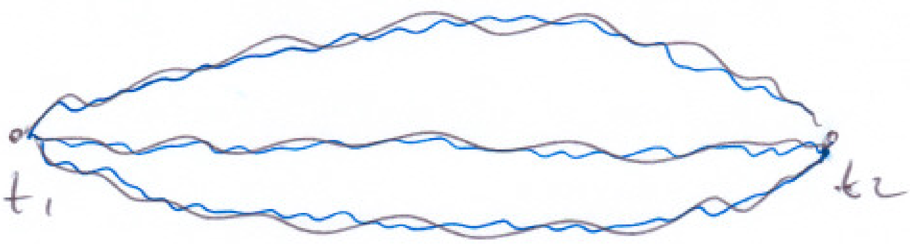
{ Can think of as limit of countable D.O.F. (as in a space-time lattice) }

Fundamental quantity is S: the action

$$\boxed{S = \int L\,dt = \int \mathcal{L}(\varphi, \partial_\mu \varphi)\, d^4x}$$

$\mathcal{L}$ is Lagrange density

"Principle of least action."



when system evolves from one configuration to another between t_1 and t_2 it does so along path for which S is an extremum.

(usually minimum)

$$0 = \delta S$$

$$= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta(\partial_\mu \varphi) \right)$$

$$= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \delta\varphi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta\varphi \right) \right)$$

Surface integral over S.T. boundary

A) $\delta\varphi$ is 0 at temporal boundaries as have initial + final field configurations.

B) Restrict ourselves to $\delta\varphi$ that vanish on spatial boundary

$\Downarrow$

Surface term vanishes

As $\delta\varphi$ is arbitrary,

$$\boxed{\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)}\right) - \frac{\partial \mathcal{L}}{\partial\varphi} = 0}$$

Euler-Lagrange E.O.M.

(1 such equation for each field.)

EXAMPLE: K-G field

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)} = \partial^\mu\varphi \qquad \frac{\partial \mathcal{L}}{\partial\varphi} = -m^2\varphi$$

E.O.M. KLEIN-GORDON EQUATION

$$\partial_\mu \partial^\mu \varphi + m^2 \varphi = 0$$

or

$$(\Box + m^2)\varphi = 0$$

(Relativistic version of S.H.O. equation)

Hamiltonian Mechanics

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \qquad \{q_i, p_j\} = \delta_{ij}$$

$$H = \sum_i p_i \dot{q}_i - L$$

Hamiltonian Field Theory

Lagrangian formulation well suited to relativity as is manifestly Lorentz invariant.

Hamiltonian formulation is more intuitive as related to non-relativistic Q.M.

But $\boxed{H \sim E \underset{\text{BOOSTS}}{\longrightarrow} E' \neq E}$

Lose manifest Lorentz invariance!

Nevertheless we will follow the Hamiltonian method.

By analogy w/ mechanics, assume

$$\pi(x^i) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(x^i)}$$

$$\boxed{H = \int d^3x \, \mathcal{H} = \int d^3x \left[\pi(x^i) \dot{\varphi}(x^i) - \mathcal{L} \right]}$$

$\mathcal{H}$ is Hamiltonian density

EXAMPLE: K-G field

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2$$

$$= \frac{1}{2} (\partial_\mu \varphi)^2 - \frac{1}{2} m^2 \varphi^2$$

$$\Rightarrow \quad \mathcal{L} = \tfrac{1}{2}\dot{\varphi}^2 - \tfrac{1}{2}(\nabla_i \varphi)^2 - \tfrac{1}{2}m^2\varphi^2$$

$$\pi(x^i) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \dot{\varphi}(x^i)$$

$$H = \int d^3x \, [\, \dot{\varphi}^2 - \tfrac{1}{2}\dot{\varphi}^2 + \tfrac{1}{2}(\nabla_i \varphi)^2 + \tfrac{1}{2}m^2\varphi^2)$$

$$\Downarrow$$

$$H = \int d^3x \, [\, \tfrac{1}{2}\dot{\varphi}^2 + \tfrac{1}{2}(\nabla_i \varphi)^2 + \tfrac{1}{2}m^2\varphi^2]$$

$$\boxed{H = \int d^3x \, [\, \tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\nabla_i \varphi)^2 + \tfrac{1}{2}m^2\varphi^2]}$$

Energy cost: moving in time; moving in space; existing at all

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Noether's Thm

Relation between symmetries + conservation laws

Consider infinitesimal transformations on φ.

$$(*) \quad \boxed{\varphi(x) \to \varphi'(x) = \varphi(x) + \alpha \, \Delta\varphi(x)}$$

infinitesimal parameter

Deformation of field

This transformation is <u>symmetry</u> if the E-L E.O.M. are left invariant.

$$\Updownarrow$$

S invariant

Hence, w/ respect to $(*)$,

$$\boxed{\mathcal{L}(x) \to \mathcal{L}(x) + \alpha \, \partial_\mu J^\mu(x)}$$

surface term, (vanishes at boundary of S.T.)

($J^\mu(x)$ is fn. of fields)

Compare this with direct variation of $\mathcal{L}$

$$\boxed{\mathcal{L}(x) \to \mathcal{L}(x) + \alpha \, \Delta\mathcal{L}}$$

$$\alpha \, \Delta \mathcal{L} = \alpha \left(\frac{\partial \mathcal{L}}{\partial \varphi} \Delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \Delta(\partial_\mu \varphi) \right)$$

$$= \frac{\partial \mathcal{L}}{\partial \varphi} (\alpha \Delta\varphi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\mu (\alpha \Delta \varphi)$$

$$= \alpha \, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \Delta\varphi \right) + \alpha \underbrace{\left[\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \right]}_{\text{E-L E.O.M.} \Rightarrow 0} \Delta\varphi$$

Comparing w/ previous $\Rightarrow$

$$\partial_\mu J^\mu(x) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \Delta\varphi \right)$$

$\Downarrow$ solution

$$\boxed{J^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \Delta\varphi - j^\mu(x)}$$

where $\boxed{\partial_\mu j^\mu(x) = 0}$

"current" $j^\mu(x)$ is conserved !

(Invert: $j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \Delta\varphi - J^\mu$)

$\therefore$ For each continuous symmetry of $\mathcal{L}$

$\Updownarrow$

conservation Law

(Noether's Thm)

(Clear generalization to more fields)

Note: Conservation law $\Rightarrow$ conserved charge

$$Q = \int j^0(x)\, d^3x$$

$$\frac{\partial Q}{\partial t} = \int \partial_0 j^0(x)\, d^3x$$

$$= \int \partial_\mu j^\mu(x)\, d^3x + \text{surface terms}$$

$$\Rightarrow \boxed{\frac{\partial Q}{\partial t} = 0}$$

(Not on as solid ground!)

Examples:

(A) $$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2$$

Invariance: $\varphi \to \varphi + \alpha$ $\quad$ α constant

Recall:

$$j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \Delta\varphi - J^\mu$$

(not relevant here)

$$j^\mu = \partial^\mu \varphi \qquad \partial_\mu j^\mu = \Box \varphi = 0$$

(K-G equation)

(B) $$\mathcal{L} = |\partial_\mu \varphi|^2 - m^2 |\varphi|^2$$

$$= \partial_\mu \varphi \, \partial^\mu \varphi^* - m^2 \varphi \varphi^*$$

φ is complex valued field

K-G equation?

treat φ, φ^* as independent fields

φ: $\quad \partial_\mu(\partial^\mu \varphi^*) + m^2 \varphi^* = 0 \Rightarrow (\Box + m^2)\varphi^* = 0$

φ^*: $\quad (\Box + m^2)\varphi = 0$

Invariance of $\mathcal{L}$? $\qquad \varphi \to e^{i\alpha}\varphi$

"phase transformation"

infinitesimal: $\quad \varphi \longrightarrow \varphi + i\alpha\varphi + \mathcal{O}(\alpha^2)$

$$\boxed{\begin{aligned} \alpha\,\Delta\varphi &= i\alpha\varphi \\ \alpha\,\Delta\varphi^* &= -i\alpha\varphi^* \end{aligned}}$$

Conserved current?

$$j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\Delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^*)}\Delta\varphi^*$$

$$= (\partial^\mu \varphi^*)(i\varphi) + (\partial^\mu \varphi)(-i\varphi^*)$$

$$\Rightarrow \boxed{j^\mu(x) = i\left[(\partial^\mu \varphi^*)\varphi - \varphi^* \partial^\mu \varphi\right]}$$

Conserved?

$$\partial_\mu j^\mu = i[(\Box\varphi^*)\varphi + \partial^\mu\varphi^*\partial_\mu\varphi - \partial_\mu\varphi^*\partial^\mu\varphi - \varphi^*(\Box\varphi)]$$

(use K-G)

$$= i[-m^2|\varphi|^2 + m^2|\varphi|^2] = 0$$

Noether's Thm also applies to

Space-time transformations

Consider S-T translations:

$$x^\mu \to x^\mu - a^\mu \qquad (a^\mu \text{ is constant})$$

φ: $$\varphi(x) \to \varphi(x+a) = \varphi(x) + a^\mu\partial_\mu\varphi(x) + O(a^2)$$

$\mathcal{L}$: $$\mathcal{L} \to \mathcal{L} + a^\mu\partial_\mu\mathcal{L} = \mathcal{L} + a^\nu\partial_\mu(\delta^\mu{}_\nu\mathcal{L})$$

4 seperately conserved currents:

$$\boxed{T^\mu{}_\nu \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\nu\varphi - \mathcal{L}\delta^\mu{}_\nu}$$

$T^{\mu}{}_{\nu}$: Energy - Momentum Tensor

Let's look more carefully at δS w/ respect to S-T translations:

$$\mathcal{L} = \mathcal{L}(\varphi, \partial_\mu \varphi, x^\mu)$$

$$x^\mu \to x^{\mu'} = x^\mu + \delta x^\mu$$

$$\delta S = \int d^4x' \, \mathcal{L}(\varphi', \partial_\mu \varphi', x^{\mu'}) - \int d^4x \, \mathcal{L}(\varphi, \partial_\mu \varphi, x^\mu)$$

$$d^4x' = J\left(\frac{x'}{x}\right) d^4x \qquad (J \text{ is Jacobian})$$

$$\frac{\partial x'^\mu}{\partial x^\lambda} = \delta^\mu{}_\lambda + \partial_\lambda \delta x^\mu$$

$$J\left(\frac{x'}{x}\right) = \det\left(\frac{\partial x'^\mu}{\partial x^\lambda}\right) = 1 + \partial_\mu(\delta x^\mu)$$

$$\Rightarrow \delta S = \int d^4x \left(\delta \mathcal{L} + \mathcal{L} \, \partial_\mu \delta x^\mu\right)$$

$$\left(\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta(\partial_\mu \varphi) + \frac{\partial \mathcal{L}}{\partial x^\mu} \delta x^\mu\right)$$

So $$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \varphi}\delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\delta(\partial_\mu \varphi) + \partial_\mu(\mathcal{L}\,\delta x^\mu)\right)$$

Source of extra piece!

<u>Conservation Laws</u>

$$T^{00} = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\dot{\varphi} - \mathcal{L}$$

$$= \pi(x)\,\dot{\varphi}(x) - \mathcal{L} = \mathcal{H}$$

So $$H = \int d^3x\, T^{00}$$

H is generator of <u>time translations</u>

$$T^0{}_i = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\partial_i \varphi = \pi\,\partial_i\varphi$$

So $$P^i = \int d^3x\, T^{0i} = -\int d^3x\, \pi\,\partial_i\varphi$$

Momenta (carried by field) generator of <u>space translations</u>