Before we address issue of renormalization let's introduce some more notation.

Define <u>self energy</u> as

$\frac{1}{i}\Sigma(p) \equiv$

external lines <u>amputated</u>

Then complete propagator is:

$$G^{(2)}(p) = G_0(p) + G_0(p)\frac{\Sigma(p)}{i}G_0(p) + G_0(p)\frac{\Sigma}{i}G_0(p)\frac{\Sigma}{i}G_0(p) + \ldots$$

$$\boxed{G^{(2)}(p) = \frac{i}{p^2 - m^2 - \Sigma(p)}} \qquad \left(G_0(p) = \frac{i}{p^2-m^2} \right)$$

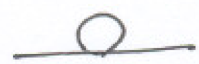
We can also define a 2-point <u>vertex function</u> $\Gamma^{(2)}$:

$$G^{(2)}(p)\,\Gamma^{(2)}(p) = i$$

$$\Downarrow$$

$$\boxed{\Gamma^{(2)}(p) = p^2 - m^2 - \Sigma(p)}$$

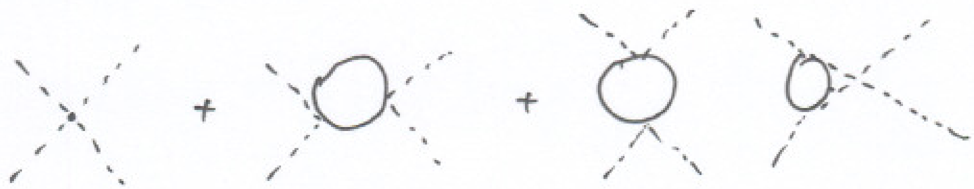
Recall that we've computed Σ !!

$$\frac{\Sigma}{i} = \frac{i\lambda m^2}{16\pi^2\epsilon} + \text{finite}$$ (= )

$$\Rightarrow \boxed{\Sigma = -\frac{\lambda m^2}{16\pi^2\epsilon} + \text{finite}}$$

$$\therefore \boxed{\Gamma^{(2)}(p) = p^2 - m^2\left(1 - \frac{\lambda}{16\pi^2\epsilon}\right)} + \text{finite}$$

Similarly, we can define a 4-point function which in our case diagrammatically is:

$\Gamma^{(4)}(p_i) =$ 

(s-channel) (t-channel) (u-channel)

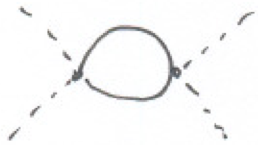
Recall 

all momenta going in.

$$\Rightarrow \quad s = (p_1+p_2)^2, \quad t = (p_1+p_3)^2, \quad u = (p_1+p_4)^2$$

 $= -i\lambda\mu^{\epsilon}$ From Feynman Rule

And we computed:



$$= \frac{i\lambda^2\mu^\epsilon}{16\pi^2\epsilon} - \frac{i\lambda^2\mu^\epsilon}{32\pi^2}\left[\gamma + F(s,m,\mu)\right]$$

$\therefore$

$$\Gamma^{(4)}(p_i) = -i\lambda\mu^\epsilon + 3i\frac{\lambda^2\mu^\epsilon}{16\pi^2\epsilon}$$

$$-i\frac{\lambda^2\mu^\epsilon}{32\pi^2}\left[3\gamma + F(s,m,\mu) + F(t,m,\mu) + F(u,m,\mu)\right]$$

$$\Rightarrow \boxed{\Gamma^{(4)}(p_i) = -i\lambda\mu^\epsilon\left(1 - \frac{3\lambda}{16\pi^2\epsilon}\right)} + \text{finite}$$

Now in order that $\lambda\varphi^4$ be a sensible QFT as defined in perturbation theory, $\Gamma^{(2)}$ and $\Gamma^{(4)}$ should be finite.

Note infinite parts in $\Gamma^{(2)}$ and $\Gamma^{(4)}$ are $\mathcal{O}(\lambda)$ and $\mathcal{O}(\lambda^2)$, respectively. However, both are 1-loop effects:

$$\boxed{\text{loop expansion} = \text{expansion in } \hbar}$$

Renormalization of $\lambda\varphi^4$

We will describe **2 ways** of thinking about

Renormalization

Method 1

Say $\Gamma^{(2)}(p) = p^2 - m^2\left(1 - \frac{\lambda}{16\pi^2\epsilon}\right) = p^2 - m_1^2$

$$\boxed{m_1^2 \equiv m^2\left(1 - \frac{\lambda}{16\pi^2\epsilon}\right)}$$

finite — physical mass

infinite — no physical significance (would be only in absence of interaction)

$$m^2 = m_1^2 + \frac{m^2\lambda}{16\pi^2\epsilon} \Rightarrow \boxed{m^2 = m_1^2\left(1 + \frac{\lambda}{16\pi^2\epsilon}\right)}$$

(shifted terms of $O(\lambda^2)$)

Note that physical or renormalized mass m_1 is

$$m_1^2 = -\Gamma^{(2)}(0)$$

Next consider $\Gamma^{(4)}$.

Recall:

(1) $$\boxed{i\Gamma^{(4)}(p_i) = \lambda\mu^{\epsilon} - \frac{\lambda^2\mu^{\epsilon}}{32\pi^2}\left[\frac{6}{\epsilon} - 3\gamma - F(s,m,\mu) - F(t,m,\mu) - F(u,m,\mu)\right]}$$

Now define a renormalized coupling λ_1 as

(*) $$\lambda_1 = \lambda\mu^{\epsilon} - \frac{\lambda^2\mu^{\epsilon}}{32\pi^2}\left[\frac{6}{\epsilon} - 3\gamma - 3F(0,m,\mu)\right]$$

Solving for λ gives:

(2) $$\boxed{\lambda = \lambda_1\mu^{-\epsilon} + \frac{3\lambda_1^2\mu^{-2\epsilon}}{32\pi^2}\left[\frac{2}{\epsilon} - \gamma - F(0,m_1,\mu)\right]}$$

(again skipped terms of $O(\lambda^3)$)

Again, λ_1 is finite and physical,

λ is infinite and unphysical

We can now plug (2) into (1):

note physical mass m_1

$$\boxed{i\Gamma^{(4)}(p_i) = \lambda_1 + \frac{\lambda_1^2\mu^{-\epsilon}}{32\pi^2}\left[F(s,m_1,\mu) + F(t,m_1,\mu) + F(u,m_1,\mu) - 3F(0,m_1,\mu)\right]}$$

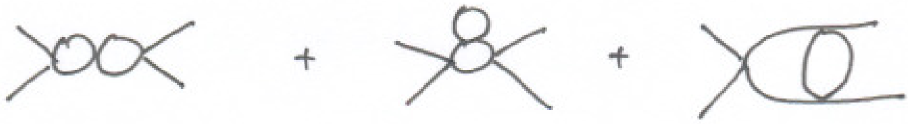
$$\boxed{i\Gamma^{(4)}(p_i=0) = \lambda_1}$$

follows from definition (*)

Now we've gotten rid of divergences at 1-loop!!

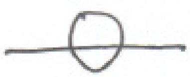
$\boxed{\text{what happens at 2-loops??}}$

$\Gamma^{(2)} \sim$ 

$\Gamma^{(4)} \sim$ 

It turns out that $\Gamma^{(4)}$ is finite at 2 loops after accounting for mass and coupling constant renormalization.

However $\Gamma^{(2)}$ has new divergence from:

 "setting sun" diagram.

New divergence is removed by "multiplicative" factor:

$$\boxed{\Gamma_r^{(2)} = Z_\phi(\lambda_1, m_1, \mu)\, \Gamma^{(2)}(p, m_1, \mu)}$$

finite

infinite

infinite at 2 loops!!

WAVE-FUNCTION RENORMALIZATION

We can expand Z_ϕ in # of loops:

$$Z_\phi = 1 + \lambda_1^2 z_2 + \dots$$

begins at 2-loops

Recall that $\Gamma_0^{(2)} = p^2 - m^2$

Comes from kinetic operator in $\mathcal{L}$:

$$\varphi(p)\,(p^2 - m^2 + i\epsilon)\,\varphi(p) = \varphi(p)\,\Gamma_0^{(2)}(p)\,\varphi(p)$$

$\therefore$ in interacting case $Z_\varphi^{\frac{1}{2}}$ is renormalization of the field amplitude:

$$\boxed{\varphi \to Z_\varphi^{-\frac{1}{2}}\,\varphi}$$

We noted that m_1 is infinite at 2 loops. However, we now have

$$\Gamma_r^{(2)}(0) = Z_\varphi\,\Gamma^{(2)}(0)$$

$$\Rightarrow \boxed{m_r^2 = Z_\varphi\, m_1^2}$$

divergences cancel

Similarly,
$$\boxed{\Gamma_r^{(n)} = Z_\varphi^{2}\,\Gamma^{(n)}(p, m, \mu)}$$

$$i\,\Gamma_r^{(4)}(p_i = 0) = \boxed{\lambda_r = Z_\varphi^2\,\lambda_1}$$

$\{$ Note that $Z_\varphi = Z_\varphi(\lambda_1 = \lambda\mu^\epsilon)$ $\}$

∴ we can write for renormalized n-particle vertex function:

$$\boxed{\Gamma_r^{(n)}(p_i, \lambda_r, m_r, \mu) = Z_\phi^{n/2}(\lambda\mu^\epsilon)\,\Gamma^{(n)}(p_i, \lambda, m)}$$

Now $\lambda\phi^4$ theory is finite to 2-loops!!

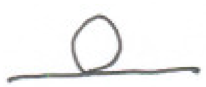
What about higher orders?? very hard to prove...

Method 2

viewpoint: parameters m and λ in original Lagrangian are physical parameters. As this $\mathcal{L}$ does not give finite Green's functions, the Lagrangian must be augmented by extra terms $\Downarrow$
to cancel the divergences.

"COUNTERTERMS"

Recall:

 $= i\dfrac{\lambda m^2}{16\pi^2\epsilon} + \text{finite}$

1-loop modification of free propagator

∴ add term to $\mathcal{L}$:

$$\boxed{\delta\mathcal{L}_1 = -\frac{\lambda m^2}{32\pi^2\epsilon}\varphi^2 = -\frac{\delta m^2}{2}\varphi^2}$$

⇒ Additional Feynman Rule:

 $= -i\frac{\lambda m^2}{16\pi^2\epsilon} = -i\,\delta m^2$

Then $\frac{1}{i}\Sigma =$  $+$ 

$$\Gamma^{(2)}(p) = p^2 - m^2 - \Sigma = p^2 - m^2 - i\left(\frac{i\lambda m^2}{16\pi^2}\frac{1}{\epsilon} + \text{finite}\right) - \frac{\lambda m^2}{16\pi^2}\frac{1}{\epsilon}$$

$$\Rightarrow \boxed{\Gamma^{(2)}(p) = p^2 - m^2}$$

(ignoring finite piece)

Here m^2 is finite physical quantity.

Similarly, for $\Gamma^{(4)}$ we can add counterterm:

$$\boxed{\delta\mathcal{L}_2 = -\frac{1}{4!}\,\frac{3\lambda^2\mu^\epsilon}{16\pi^2\epsilon}\varphi^4 = -\frac{B\,\lambda\mu^\epsilon}{4!}\varphi^4}$$

 $= -\frac{3i\lambda^2\mu^\epsilon}{16\pi^2\epsilon}$ F.R.

Then

$\Gamma^{(4)}(p_i) =$  +  + CROSSED + 

$$\boxed{\Gamma^{(4)}(p_i) = -i\lambda\mu^{\epsilon} + \text{finite}}$$

✓

Wave function renormalization ??

$$\boxed{\delta\mathcal{L}_3 = \frac{A}{2}(\partial_\mu\varphi)^2} \qquad (1 + A = Z_\varphi)$$

Hence, finite correlation functions may be obtained from

$$\boxed{\begin{gathered}\mathcal{L} = \tfrac{1}{2}(\partial_\mu\varphi)^2 - \frac{m^2}{2}\varphi^2 - \frac{\lambda\mu^{\epsilon}}{4!}\varphi^4 + \mathcal{L}_{CT} \\ \text{where } \mathcal{L}_{CT} = \frac{A}{2}(\partial_\mu\varphi)^2 - \frac{\delta m^2}{2}\varphi^2 - \frac{B\lambda\mu^{\epsilon}}{4!}\varphi^4\end{gathered}}$$

Now we can define a "Bare" Lagrangian as

$$\mathcal{L}_B = \tfrac{1}{2}(\partial_\mu\varphi_B)^2 - \frac{m_B^2}{2}\varphi_B^2 - \frac{\lambda_B}{4!}\varphi^4$$

where

$$\varphi_B = \sqrt{Z_\varphi}\,\varphi \qquad Z_\varphi = 1 + A$$

$$m_B = Z_m\, m \qquad Z_m^2 = \frac{m^2 + \delta m^2}{m^2(1+A)}$$

$$\lambda_B = \mu^\epsilon Z_\lambda \lambda \qquad Z_\lambda = \frac{1+B}{(1+A)^2}$$

In this "counterterm" viewpoint, a theory is renormalizable if counterterms required to cancel divergences are of the same form as those appearing in the original Lagrangian.

The Renormalization Group

We saw that in dim reg it was necessary to introduce a new mass parameter μ.

Recall:

$$\boxed{\Gamma_r^{(n)}(p_i, \lambda_r, m_r, \mu) = Z_\varphi^{n/2}(\lambda\mu^\epsilon)\,\Gamma^{(n)}(p_i, \lambda, m)}$$

Renormalized n-point functions depend on μ, through Z_φ

Hence, unrenormalized $\Gamma^{(n)}(p_i, \lambda, m)$ is independent of μ: i.e. invariant under the transformation:

$$\boxed{\mu \to e^s \mu}$$

Consider dimensionless differential operator $\mu \frac{\partial}{\partial \mu}$.

$$\mu \frac{\partial}{\partial \mu} \Gamma^{(n)} = 0 \qquad \text{or}$$

$$\boxed{\mu \frac{d}{d\mu} \left[Z_\phi^{-n/2}(\lambda \mu^\epsilon) \, \Gamma_r^{(n)}(p_i, \lambda_r, m_r, \mu) \right] = 0}$$

depend implicitly on μ.

$\Downarrow$ carry out differentiation

$$\mu \left[-\frac{n}{2} Z_\phi^{-n/2 - 1} \frac{\partial}{\partial \mu} Z_\phi \, \Gamma_r^{(n)} + \left(\frac{d}{d\mu} \Gamma_r^{(n)} \right) Z_\phi^{-n/2} \right] = 0$$

$$\Rightarrow \mu \left[-n \frac{\partial}{\partial \mu} \log \sqrt{Z_\phi} + \left(\frac{\partial}{\partial \mu} + \frac{\partial \lambda_r}{\partial \mu} \frac{\partial}{\partial \lambda_r} + \frac{\partial m_r}{\partial \mu} \frac{\partial}{\partial m_r} \right) \right] \Gamma_r^{(n)} = 0$$

$$\Rightarrow \boxed{\left[-n \mu \frac{\partial}{\partial \mu} \log \sqrt{Z_\phi} + \mu \frac{\partial}{\partial \mu} + \mu \frac{\partial \lambda_r}{\partial \mu} \frac{\partial}{\partial \lambda_r} + \mu \frac{\partial m_r}{\partial \mu} \frac{\partial}{\partial m_r} \right] \Gamma_r^{(n)} = 0}$$

This can be simplified by defining:

$$\boxed{\begin{aligned} \gamma(\lambda) &= \mu \frac{\partial}{\partial\mu} \log \sqrt{Z_\varphi} \\ \beta(\lambda) &= \mu \frac{\partial\lambda}{\partial\mu} \\ m\,\gamma_m(\lambda) &= \mu \frac{\partial m}{\partial\mu} \end{aligned}}$$

$$\Rightarrow \boxed{\left[\mu \frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} - n\,\gamma(\lambda) + m\gamma_m(\lambda)\frac{\partial}{\partial m}\right]\Gamma^{(n)} = 0}$$

Renormalization group equation

(invariance of $\Gamma^{(n)}$ under change of μ)

Notice that even in massless limit when $\lambda\varphi^4$ theory is scale invariant, there is still non-trivial evolution in μ as $\gamma, \beta \neq 0$.

$\Downarrow$

Quantum mechanical breaking of scale invariance