Regularizing Divergent Integrals

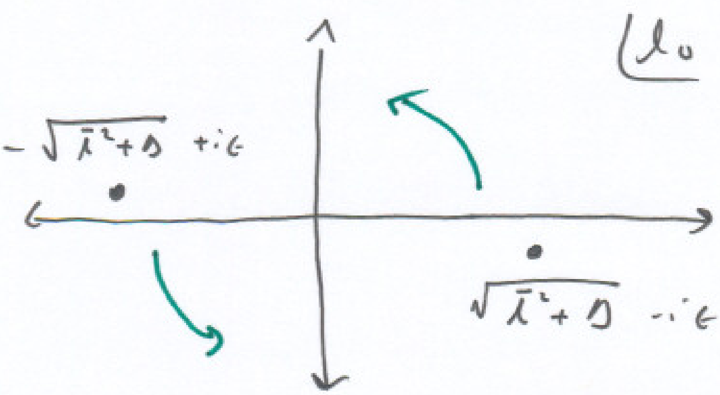
Consider:

$$I^2(\Delta) = \underbrace{\int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2-\Delta)^2}}_{\text{logarithmically divergent}} \qquad \left(\sim \int \frac{d^4 l}{l^4}\right)$$

One can use contour integrals to do l_0 integral and spatial integrals in spherical coordinates.

But can also do trick:

$\boxed{\text{Minkowski} \Longrightarrow \text{Euclidean}}$ Wick Rotation



(90° counter clockwise rotation)

Euclidean 4-momenta

$$l^0 \equiv i l_E^0 \qquad l_i = l_{iE}$$

<u>Rotated contour</u> : $l_E^0 = -\infty$ to ∞

$$I^m(\Delta) = \int \frac{d^4 l}{(2\pi)^4} \frac{1}{[l^2 - \Delta]^m} = \int dl_0\, d^3\vec{l} \; \frac{1}{[l_0^2 - \vec{l}^2 - \Delta]^m}$$

$$= \frac{i}{(-1)^m} \int d^4 l_E \frac{1}{[l_E^2 + \Delta]^m}$$ (simple change of variables)

As $I^2(\Delta)$ is logarithmically divergent, we would like to regularize it.

Consider Pauli-Villars procedure:

$$\frac{1}{p^2 - m^2} \to \frac{1}{p^2 - m^2} - \frac{1}{p^2 - \Lambda^2} = \frac{(m^2 - \Lambda^2)}{(p^2 - m^2)(p^2 - \Lambda^2)}$$

↳ violates unitarity at the scale Λ.

$$I^2(\Delta) = \frac{i}{(-1)^2} \frac{1}{(2\pi)^4} \int d^4 l_E \frac{1}{[l_E^2 + \Delta]^2}$$

$$= \frac{i}{(2\pi)^4} \int d\Omega_4 \int_0^\infty dl_E \frac{l_E^3}{[l_E^2 + \Delta]^2}$$

↳ surface area of 4-d unit sphere

$\int d\Omega_d$ = area of unit sphere in d dimensions

Evaluate using a trick:

$$(\sqrt{\pi})^d = \left(\int dx\, e^{-x^2}\right)^d = \int d^d x\, e^{-\sum_{i=1}^{d} x_i^2}$$

$$= \int d\Omega_d \int_0^\infty dx\, x^{d-1} e^{-x^2} = \left(\int d\Omega_d\right)\cdot \frac{1}{2}\int_0^\infty d(x^2)(x^2)^{\frac{d}{2}-1} e^{-(x^2)}$$

$$\int_0^\infty dy\, y^{\frac{d}{2}-1} e^{-y} = \Gamma(d/2)$$ Gamma function

$$\Rightarrow \boxed{\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}}$$

d	$\Gamma(d/2)$	$\int d\Omega_d$
1	$\sqrt{\pi}$	2
2	1	2π
3	$\sqrt{\pi}/2$	4π
4	1	$2\pi^2$

$$I^2(\Delta) = \frac{i}{(2\pi)^4}(2\pi^2)\,\frac{1}{2}\int_0^\infty d\ell_E^2\, \frac{\ell_E^2}{[\ell_E^2+\Delta]^2}$$

Now do Pauli-Villars

$$I^2_{\mu\nu}(\Delta) = \frac{i}{(4\pi)^2}\int_0^\infty d\ell_E^2 \left(\frac{\ell_E^2}{[\ell_E^2+\Delta]^2} - \frac{\ell_E^2}{[\ell_E^2+\Delta_\Lambda]^2} \right)$$

$$\Rightarrow \boxed{I^2_{\mu\nu}(\Delta) = \frac{i}{(4\pi)^2}\log\frac{\Delta_\Lambda}{\Delta}} \qquad \left(\Delta_\Lambda \xrightarrow[\Lambda\to\infty]{} \sim \Lambda^2\right)$$

Dimensional Regularization

Compute Feynman diagram in d spacetime dimensions.

For small d, loop integrals converge and are therefore regularized.

$\boxed{\text{Preserves all symmetries and physical principles!!}}$

Recall: $$I^2(\Delta) = \int \frac{d^d\ell_E}{(2\pi)^d}\frac{1}{(\ell_E^2+\Delta)^2}$$

$$= \int \frac{d\Omega_d}{(2\pi)^d}\int_0^\infty d\ell_E \frac{\ell_E^{d-1}}{(\ell_E^2+\Delta)^2}$$

$$= \frac{1}{(2\pi)^d}\cdot\frac{2\pi^{d/2}}{\Gamma(d/2)}\underbrace{\int_0^\infty d\ell \frac{\ell^{d-1}}{(\ell^2+\Delta)^2}}$$

$$\int_0^\infty d\lambda \frac{\lambda^{d-1}}{(\lambda^2+\Delta)^2} = \frac{1}{2}\int_0^\infty d(\lambda^2)\frac{(\lambda^2)^{d/2-1}}{(\lambda^2+\Delta)^2}$$

$$= \frac{1}{2}\left(\frac{1}{\Delta}\right)^{2-d/2}\int_0^1 dx\, x^{1-d/2}(1-x)^{d/2-1} \qquad \left\{\lambda^2 = \frac{\Delta}{x}+\Delta\right\}$$

Now note that

$$\int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \qquad (\equiv B(\alpha,\beta))$$

So $$I^2_{D/2}(\Delta) = \frac{1}{(2\pi)^d}\cdot\frac{2\pi^{d/2}}{\Gamma(d/2)}\frac{1}{2}\left(\frac{1}{\Delta}\right)^{2-d/2}\frac{\Gamma(2-d/2)\Gamma(d/2)}{\Gamma(2)}$$

$$\Rightarrow \boxed{I^2_{D/2}(\Delta) = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{\Gamma(2)}\left(\frac{1}{\Delta}\right)^{2-d/2}}$$

where is divergence ??

$\Gamma(z)$ has isolated poles at $z = 0, -1, -2, \ldots$

$\therefore$ $I^2_{D/2}(\Delta)$ has isolated poles at $d = 4, 6, 8, \ldots$

So $\epsilon = 4-d$

$$\boxed{\Gamma(2-d/2) = \Gamma(\epsilon/2) = \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon)}$$

$\gamma \simeq 0.5772$ Euler-Mascheroni

Now note that

$$(4\pi)^{-d/2} = (4\pi)^{-2+\frac{\epsilon}{2}} = (4\pi)^{-2} e^{\log 4\pi^{\frac{\epsilon}{2}}}$$

$$= \frac{1}{(4\pi)^2}\left(1 + \frac{\epsilon}{2}\log 4\pi + \mathcal{O}(\epsilon^2)\right)$$

$$\Delta^{-2+d/2} = \Delta^{-\epsilon/2} = e^{\log \Delta^{-\epsilon/2}} = \left(1 - \frac{\epsilon}{2}\log\Delta + \mathcal{O}(\epsilon^2)\right)$$

$$\boxed{I^2_{DR}(\Delta) \xrightarrow[d\to 4]{} \frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma - \log\Delta + \log 4\pi + \mathcal{O}(\epsilon)\right)}$$

single pole corresponds to log divergence of loop integral.

Note that unlike P.V., the scale of the logarithm is hidden in $1/\epsilon$ and appears explicitly only when divergence is removed (as we will see)

Dimensional Regularization of $\lambda\varphi^4$ theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4$$

First we should generalize $\mathcal{L}$ to d dimensions.

Recall $[\varphi] = \frac{d}{2} - 1$ $\qquad [\lambda] = 0$ in $d=4$

$$[\lambda] = \Lambda^{4-d}$$

$\therefore$ to keep dimensionless in d dimensions $\Rightarrow$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\mu^{4-d}}{4!}\lambda\phi^4$$

μ = arbitrary mass scale

Consider our first divergent graph $\sim D_F(0)$

$$= \frac{1}{2}\lambda\mu^{4-d}\int\frac{d^dp}{(2\pi)^d}\frac{1}{(p^2-m^2)}$$

{Symmetry factor}

$$= -\frac{i\lambda}{32\pi^2}\, m^2 \left(\frac{4\pi\mu^2}{-m^2}\right)^{2-d/2}\Gamma\left(1-\frac{d}{2}\right)$$

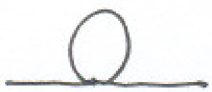
(See Ryder for direct Minkowski space evaluation)

Generally,

$$\boxed{\begin{aligned}\Gamma(-n+\epsilon) &= \frac{(-1)^n}{n!}\left[\frac{1}{\epsilon} + \psi_1(n+1) + O(\epsilon)\right]\\ \psi_1(n+1) &= 1 + \frac{1}{2} + \dots + \frac{1}{n} - \gamma\end{aligned}}$$

Again setting $\underline{\epsilon = 4-d}$ $\Rightarrow$

$$\Gamma(1-d/2) = \Gamma(-1+\tfrac{\epsilon}{2}) = -\frac{2}{\epsilon} - 1 + \gamma + O(\epsilon)$$



$$= -\frac{i\lambda m^2}{32\pi^2}\left[-\frac{2}{\epsilon} - 1 + \gamma + O(\epsilon)\right]\left[1 + \frac{\epsilon}{2}\log\left(\frac{4\pi\mu^2}{-m^2}\right)\right]$$

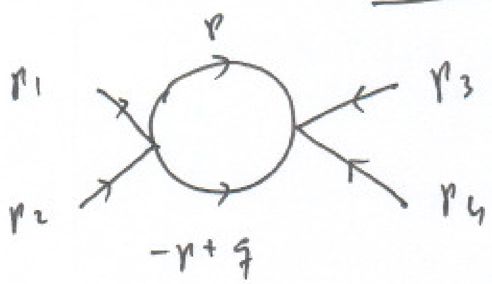
{Again used: $x^\epsilon = 1 + \epsilon \log x$ }

$$= \frac{i\lambda m^2}{16\pi^2\epsilon} + \frac{i\lambda m^2}{32\pi^2}\left[1 - \gamma + \log\left(\frac{4\pi\mu^2}{-m^2}\right)\right] + O(\epsilon)$$

$$= \frac{i\lambda m^2}{16\pi^2\epsilon} + \underline{\text{finite}}$$

Our <u>second</u> divergent graph is from scattering:

In the <u>s-channel</u>: $(p_1+p_2)^2 = q^2 = s$



$$= \frac{1}{2}\lambda^2(\mu^2)^{4-d}\int \frac{d^dp}{(2\pi)^d}\frac{1}{(p^2-m^2)}\frac{1}{[(p-q)^2-m^2]}$$

Now can use another trick: <u>Feynman parametrization</u>

$$\frac{1}{ab} = \int_0^1 \frac{dz}{[az+(1-z)b]^2}$$

(You can check that this makes sense)

$$\frac{1}{p^2-m^2}\,\frac{1}{(p-q)^2-m^2} = \int_0^1 \frac{dz}{\left[p^2-m^2-2pq(1-z)+q^2(1-z)^2\right]^2}$$

change variables: $p' = p - q(1-z)$

$$\Rightarrow \quad \frac{1}{2}\lambda^2(\mu^2)^{4-d}\int_0^1 dz \int \frac{d^dp}{(2\pi)^d}\,\frac{1}{\left[p^2-m^2+q^2z(1-z)\right]^2}$$

(mixed term is gone!!)

This integral is of form $I^2(\omega)$!!

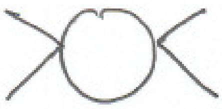
$$= \frac{i\lambda^2}{32\pi^2}(\mu^2)^{2-d/2}\,\Gamma(2-d/2)\int_0^1 dz\left[\frac{q^2z(1-z)-m^2}{4\pi\mu^2}\right]^{\frac{d}{2}-2}$$

$$= \frac{i\lambda^2\mu^\epsilon}{32\pi^2}\left(\frac{2}{\epsilon}-\gamma+O(\epsilon)\right)\left\{1-\frac{\epsilon}{2}\int_0^1 dz\log\left[\frac{q^2z(1-z)-m^2}{4\pi\mu^2}\right]\right\}$$

$$= \frac{i\lambda^2\mu^\epsilon}{16\pi^2\epsilon} - \frac{i\lambda^2\mu^\epsilon}{32\pi^2}\left\{\gamma+\int_0^1 dz\log\left[\frac{q^2z(1-z)-m^2}{4\pi\mu^2}\right]\right\}$$

Define $F(s,m,\mu) \equiv \int_0^1 dz\log\left[\frac{sz(1-z)-m^2}{4\pi\mu^2}\right]$

So finally we have:



$$= \frac{i\lambda^2\mu^{\epsilon}}{16\pi^2\epsilon} - \frac{i\lambda^2\mu^{\epsilon}}{32\pi^2}\left[\gamma + F(s,m,\mu)\right]$$

$$= \frac{i\lambda^2\mu^{\epsilon}}{16\pi^2\epsilon} + \underline{\text{finite}}$$

So we have obtained in explicit form the lowest-order corrections to 2-point and 4-point function of $\lambda\phi^4$ theory.