Basic object in Functional Quantization:

$$\boxed{Z[J] = \int \mathcal{D}\varphi \; \exp\left[ i \int d^4x \left[ \mathcal{L} + J(x)\varphi(x) \right] \right]}$$

SOURCE TERM

Note:
$$\frac{\delta}{\delta J(x)} \exp\left[ i \int d^4y \; J(y)\varphi(y) \right] = i\varphi(x) \exp\left[ i \int d^4y \, J(y)\varphi(y) \right]$$

$\therefore$

$$\langle 0 | T\{\varphi(x_1)\varphi(x_2)\} | 0 \rangle = \frac{\int \mathcal{D}\varphi \, \varphi(x_1)\varphi(x_2) \exp\left[ i \int d^4x \, \mathcal{L} \right]}{\int \mathcal{D}\varphi \; \exp\left[ i \int d^4x \, \mathcal{L} \right]}$$

$$= \frac{1}{Z[0]} \left( -i \frac{\delta}{\delta J(x_1)} \right) \left( -i \frac{\delta}{\delta J(x_2)} \right) Z[J] \Big|_{J=0}$$

In free theory $Z[J]$ can be written in explicit form.

Note: $\frac{1}{2} \partial_\mu \varphi \, \partial^\mu \varphi = -\frac{1}{2} \varphi \, \Box \, \varphi$ + total derivative

$\therefore$

$$\int d^4x \, [\mathcal{L}_0(\varphi) + J\varphi] = \int d^4x \, [\tfrac{1}{2}\varphi(-\Box - m^2 + i\epsilon)\varphi + J\varphi]$$

Feynman prescription = convergence factor for path integral

$\therefore$

$$\circledast \quad \boxed{Z_0[J] = \int D\varphi \exp\left\{-i\int [\tfrac{1}{2}\varphi(\Box + m^2 - i\epsilon)\varphi - \varphi J]\, d^4x\right\}}$$

Generating functional for free scalar field.

<u>Note</u>: $\varphi$ does <u>not</u> obey K.G. equation.

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Let's evaluate $\circledast$:

Say $\underline{\varphi \to \varphi + \varphi_0}$

$$\int [\tfrac{1}{2}\varphi(\Box + m^2 - i\epsilon)\varphi - \varphi J]\, d^4x \longrightarrow$$

$$\int [\tfrac{1}{2}\varphi(\Box + m^2 - i\epsilon)\varphi + \varphi(\Box + m^2 - i\epsilon)\varphi_0$$
$$+ \tfrac{1}{2}\varphi_0(\Box + m^2 - i\epsilon)\varphi_0 - \varphi J - \varphi_0 J]\, d^4x$$

$$\left\{ \text{Used} \int \varphi \Box \varphi_0 = \int \varphi_0 \Box \varphi \right\}$$

Now choose $\varphi_0$ such that:

$$(\Box + m^2 - i\epsilon)\, \varphi_0(x) = J(x) \qquad \boxed{*}$$

Then $$Z_0[J] = \int D\varphi \, \exp\left\{-i\int \left[\tfrac{1}{2}\varphi(\Box + m^2 - i\epsilon)\varphi - \tfrac{1}{2}\varphi_0 J\right] d^4x\right\}$$

But we know solution to $\boxed{*}$ !!

$\left\{ \text{Recall that} \quad (\Box + m^2 - i\epsilon)\, D_F(x) = -i\delta^4(x) \right\}$

$$\Rightarrow \qquad \varphi_0(x) = -i\int D_F(x-y)\, J(y)\, d^4y$$

Plugging in above we have:

$$\boxed{Z_0[J] = Z_0[0]\, \exp\left[-\frac{1}{2}\int J(x)\, D_F(x-y)\, J(y)\, d^4x\, d^4y\right]}$$

Normalization

Now we can compute the 2-pt. function.

$$\langle 0 | T\{\varphi(x_1)\varphi(x_2)\} | 0 \rangle = \frac{1}{Z_0[0]} \left(\frac{1}{i}\frac{\delta}{\delta J(x_1)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J(x_2)}\right) Z_0[J] \Big|_{J=0}$$

$$= -\frac{\delta}{\delta J(x_1)}\left[-\frac{1}{2}\int d^4y\, D_F(x_2-y)\, J(y) - \frac{1}{2}\int d^4x\, J(x)\, D_F(x-x_2)\right]$$

$$\times \frac{Z[J]}{Z_0}\Big|_{J=0}$$

$$\Rightarrow \langle 0|T\{\varphi(x_1)\varphi(x_2)\}|0\rangle = D_F(x_1 - x_2)$$

Note that $\langle 0|T\{\varphi(x)\}|0\rangle = \langle 0|\varphi(x)|0\rangle$

$$= \frac{1}{Z_0(0)} \frac{1}{i} \frac{\delta}{\delta J(x)} Z(J)\Big|_{J=0} = 0$$

To evaluate higher n-point functions adopt compact notation:

$$\int d^4x\, d^4y\, J(x) D_F(x-y) J(y) \equiv J_x D_{xy} J_y$$

↑ SUMMATION CONVENTION

$$\langle 0|T\{\varphi_1\varphi_2\varphi_3\varphi_4\}|0\rangle = \frac{1}{Z_0(0)} \frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\frac{\delta}{\delta J_3}\frac{\delta}{\delta J_4} e^{-iJ_x D_{xy} J_y}\Big|_{J=0}$$

$$= \frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\frac{\delta}{\delta J_3}\left[-J_x D_{x4}\right] e^{-iJ_x D_{xy} J_y}\Big|_{J=0}$$

$$= \frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\left[-D_{34} + J_x D_{x4} J_y D_{y3}\right] e^{-iJ_x D_{xy} J_y}\Big|_{J=0}$$

$$= \frac{\delta}{\delta J_1}\left( D_{34} J_x D_{x2} + D_{24} J_y D_{y3} + J_x D_{x4} D_{23}\right) e^{-iJ_x D_{xy} J_y}\Big|_{J=0}$$

$$\left\{ \langle 0|T\{\varphi_1\varphi_2\varphi_3\}|0\rangle = 0 \right\}$$

$$= D_{34} D_{12} + D_{24} D_{13} + D_{14} D_{23}$$

# Renormalization

Our approach to QFT is based on perturbation theory. As we will see, beyond leading order there are divergences that must be dealt with in order to define QFT.

Recall: $\lambda \varphi^4$ Theory

$$\sim \lambda \, D_F(x-x)$$

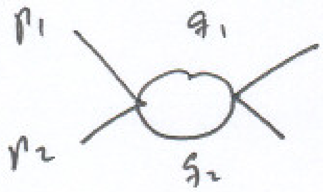
$$\sim \lambda \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2 - m^2} \qquad \left( \sim \int \frac{d^4 q}{q^2} \right)$$

Has 4 powers of $q$ in numerator, 2 powers in denominator

$\Rightarrow$ quadratic divergence

Example of an ultra-violet divergence

Another example at $\mathcal{O}(\lambda^2)$:




$$\sim \lambda^2 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{\delta(q_1 + q_2 - p_1 - p_2)}{(q_1^2 - m^2)(q_2^2 - m^2)}$$

$$\sim \lambda^2 \int \frac{d^4 q}{(2\pi)^8} \frac{1}{(q^2 - m^2)\left[(p_1 + p_2 - q)^2 - m^2\right]}$$

$\left( \sim \int \frac{d^4 q}{q^4} \right)$ logarithmic divergence

How do we find degree of divergence of a generic graph??

propagator: $q^{-2}$

vertex: $q^4 \times \delta$-fn.

loops: # of independent momenta over which integration takes place.

Consider diagram with:

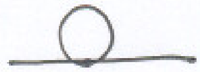
- $n$ vertices
- $E$ external lines
- $I$ internal lines
- $L$ loops

$d$ space-time dimension

Superficial Degree of Divergence: $D$

$$\boxed{D = dL - 2I}$$

Check:

 $\Rightarrow D = 4 \cdot 1 - 2 \cdot 1 = 2$ ✓

 $\Rightarrow D = 4 \cdot 1 - 2 \cdot 2 = 0$ ✓

Now would like to express $D$ in terms of $E$ and $n$.

(Must eliminate $I$ and $L$.)

There are $I$ internal momenta and momentum conservation at each vertex + overall momentum conservation:

$$\boxed{\#\text{ of indep}^{\text{nt}}\text{ momenta} = L = I - (n-1)} \quad (1)$$

In $\lambda\varphi^4$ theory there are $4n$ legs

(4 at each vertex)

$$\boxed{4n = E + 2I} \quad (2)$$

↑ internal ones count twice

So $D = dL - 2I$

$= d(I - n + 1) - 2I$ (1)

$= d(2n - \frac{E}{2} - n + 1) - (4n - E)$ (2)

$$\boxed{D = d - \left(\frac{d}{2} - 1\right)E + n(d-4)}$$

For $d=4$ we have

$$\boxed{D = 4 - E}$$

check:

 $\Rightarrow D = 4 - 2 = 2$ ✓

 $\Rightarrow D = 4 - 4 = 0$ ✓

This indicates that all diagrams w/ $\underline{E > 4}$ will converge !!

As $D$ depends on $E$ only in $d=4$, there is only small number of divergent graphs and we hope that these effects can be eliminated by an infinite Renormalization of various physical quantities

If this is true, theory is said to be:

Renormalizable

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Aside: Consider $\lambda \varphi^r$ theory:

$$\boxed{rn = E + 2I}$$

$$\Downarrow$$

$$\boxed{D = d - \left(\frac{d}{2} - 1\right)E + n\left[\frac{r}{2}(d-2) - d\right]}$$

For $d=4$ we have

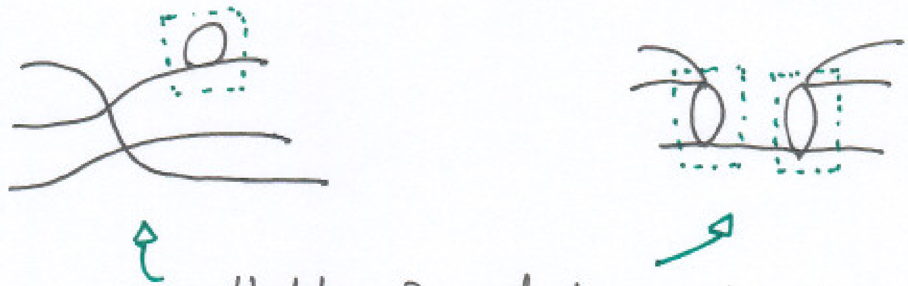
$$\boxed{D = 4 - E + n(r-4)}$$

$\lambda \varphi^6 \Rightarrow$ non renormalizable $\quad (D = 4 - E + 2n)$

$\lambda \varphi^3 \Rightarrow$ "super"-renormalizable $\quad (D = 4 - E - n)$

Note:

In $\lambda\varphi^4$ theory all graphs with $E=6$ should be convergent. However consider:



Hidden 2- and 4-point divergences!!

Hence $D$ is "superficial" degree of divergence.

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Aside on dimensional Analysis

Consider the $d$-dimensional action:

$$\boxed{S = \int d^d x\, \mathcal{L}}$$

dimensionless w/ $\hbar = 1$

$L$ : length $\qquad \Lambda$ : momentum, energy

$$[\mathcal{L}] = L^{-d} = \Lambda^{d}$$

$\mathcal{L}_0 \sim \partial^\mu \varphi\, \partial_\mu \varphi$ and $[\partial_\mu] = L^{-1}$ so

$$\boxed{[\varphi] = L^{1-d/2} = \Lambda^{d/2-1}}$$

Now consider interaction: $\lambda \varphi^r$

If $[\lambda] = L^{-\delta} = \Lambda^{\delta}$ then

$$-\delta + r(1 - d/2) = -d$$

$$\Rightarrow \boxed{\delta = d + r - \frac{rd}{2}} \quad (*)$$

$$\begin{array}{lll} & & d=4 \\ \lambda\varphi^4: & [\lambda] = \Lambda^{4-d} & \Lambda^0 \\ \lambda\varphi^3: & [\lambda] = \Lambda^{3-d/2} & \Lambda \\ \lambda\varphi^6: & [\lambda] = \Lambda^{6-2d} & \Lambda^{-2} \end{array}$$

Recall:

$$\boxed{D = d - \left(\frac{d}{2} - 1\right)E + n\left[\frac{r}{2}(d-2) - d\right]}$$

(Superficial degree of divergence in $\lambda\varphi^r$ theory)

Plug in (*):

$$\boxed{D = d - \left(\frac{d}{2} - 1\right)E - n\delta}$$

Hence renormalizable theory must have $\delta \geq 0$