Path Integral for General Quantum System

"coordinates": q_i "momenta": p_i

Hamiltonian: $H(q, p)$

$\{q_i\}$ $\{p_i\}$

Let's give proof of path integral for transition amplitude:

$$\boxed{U(q_a, q_b; T) = \langle q_b | e^{-iHT} | q_a \rangle}$$

First, break up time interval:

$$e^{-iHT} = e^{-iH\epsilon} e^{-iH\epsilon} \cdots e^{-iH\epsilon}$$

(N slices of ϵ)

Next insert complete set of states between each of these factors:

$$\mathbb{1} = \left(\prod_i \int dq_k^i \right) |q_k\rangle\langle q_k|$$

$\{ k = 1 .. (N-1) \}$

So we have:

$$\langle q_b| e^{-iHT} |q_a\rangle = \langle q_b| e^{-iH\epsilon} e^{-iH\epsilon} \cdots e^{-iH\epsilon} |q_a\rangle$$

$$= \langle q_b| e^{-iH\epsilon} \underbrace{\left(\prod_i \int dq^i_{N-1}\right)|q_{N-1}\rangle\langle q_{N-1}|} e^{-iH\epsilon} \cdots$$

$$\underbrace{\left(\prod_i \int dq^i_2\right)|q_2\rangle\langle q_2|} e^{-iH\epsilon} \underbrace{\left(\prod_i \int dq^i_1\right)|q_1\rangle\langle q_1|} e^{-iH\epsilon} |q_a\rangle$$

$$= \left(\prod_i \int dq^i_{N-1}\right) \cdots \left(\prod_i \int dq^i_2\right)\left(\prod_i \int dq^i_1\right)$$

$$\times \underbrace{\langle \underset{\underset{q_N}{\parallel}}{q_b}| e^{-iH\epsilon} |q_{N-1}\rangle} \cdots \underbrace{\langle q_2| e^{-iH\epsilon} |q_1\rangle} \underbrace{\langle q_1| e^{-iH\epsilon} |\underset{\underset{q_0}{\parallel}}{q_a}\rangle}$$

We have products of:

$$\boxed{\langle q_{k+1}| e^{-iH\epsilon} |q_k\rangle \xrightarrow[\epsilon \to 0]{} \langle q_{k+1}|(\mathbb{1} - iH\epsilon + \ldots)|q_k\rangle}$$

What do we expect for $H(q,p)$??

Say $H(q,p) \sim f(q)$

$$\langle q_{k+1}| f(q) |q_k\rangle = f(q_k) \prod_i \delta(q^i_k - q^i_{k+1})$$

$$= f\left(\frac{q_{k+1}+q_k}{2}\right)\left(\prod_i \int \frac{dp^i_k}{2\pi}\right) \exp\left[i \sum_i p^i_k (q^i_{k+1} - q^i_k)\right]$$

Now say $\underline{H(q,p) \sim f(p)}$

$$\langle q_{k+1}| f(p) |q_k\rangle = \langle q_{k+1}| f(p) \underbrace{\left(\prod_i \int dp_k^i\right) |p_k\rangle\langle p_k|}_{\text{"}1\text{"}} q_k\rangle$$

$$= \left(\prod_i \int dp_k^i\right) f(p_k) \langle q_{k+1}|p_k\rangle\langle p_k|q_k\rangle$$

$$\propto \left(\prod_i \int \frac{dp_k^i}{2\pi}\right) f(p_k) \exp\left[i\sum_i p_k^i (q_{k+1}^i - q_k^i)\right]$$

So if $H(q,p)$ contains only terms of form $f(q)$, $f(p)$,

$$\boxed{\langle q_{k+1}| H(q,p)|q_k\rangle = \left(\prod_i \int \frac{dp_k^i}{2\pi}\right) H\left(\frac{q_{k+1}+q_k}{2}, p_k\right) e^{i\sum_i p_k^i(q_{k+1}^i - q_k^i)}}$$

{ In general, H will contain terms that are products of q's & p's. By commuting q's & p's we can put in above form but with some extra terms. This form is called Weyl ordered. }

Assuming H is Weyl ordered we have:

$$\boxed{\begin{aligned}\langle q_{k+1}| e^{-i\epsilon H}|q_k\rangle = &\left(\prod_i\int \frac{dp_k^i}{2\pi}\right)\exp\left(-i\epsilon H\left(\frac{q_{k+1}+q_k}{2}, p_k\right)\right)\\ &\times \exp\left[i\sum_i p_k^i (q_{k+1}^i - q_k^i)\right]\end{aligned}}$$

Putting it all together:

$$U(q_0, q_N; T) = \left(\prod_i \int dq^i_{N-1}\right) \cdots \left(\prod_i \int dq^i_1\right)$$

$$\times \langle q_N | e^{-iH\epsilon} | q_{N-1}\rangle \cdots \langle q_1 | e^{-iH\epsilon} | q_0\rangle$$

$$\Rightarrow \boxed{\begin{aligned} U(q_0, q_N; T) &= \left(\prod_{i,k} \int dq^i_k \int \frac{dp^i_k}{2\pi}\right) \\ &\times \exp\left[i \sum_k \left(\sum_i p^i_k (q^i_{k+1} - q^i_k) - \epsilon H\left(\frac{q_{k+1}+q_k}{2}, p_k\right)\right)\right] \end{aligned}}$$

$\Downarrow$ continuum limit

$$\boxed{U(q_a, q_b; T) = \left(\prod_i \int \mathcal{D}q(t)\,\mathcal{D}p(t)\right) \exp\left[i \int_0^T dt \left(\sum_i p^i \dot{q}^i - H(q^i, p^i)\right)\right]}$$

Note: only $q(t)$ are constrained at the endpoints

This is most general path integral formula.

Measure: $\prod_i \int \frac{dq^i\, dp^i}{2\pi\hbar}$ at each point in time

$\Uparrow$

standard phase space integral!

{ One can check that for non-relativistic particle w/ $H = \frac{p^2}{2m} + V(q)$ one recovers previous result }

Path Integrals in Field Theory

Consider scalar field $\varphi(x)$.

How do we get Feynman rules from path integral??

Our general path integral formula should hold.

$$\boxed{q^i \Rightarrow \varphi(x_i)}$$

Recall:

$$\boxed{H = \int d^3x \left[\tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\nabla\varphi)^2 + V(\varphi) \right]}$$

Thus,

$$\boxed{\langle \varphi_b(x_i) | e^{-iHT} | \varphi_a(x_i) \rangle = \int D\varphi\, D\pi \exp\left[i \int_0^T d^4x \left(\pi\dot{\varphi} - \tfrac{1}{2}\pi^2 - \tfrac{1}{2}(\nabla\varphi)^2 - V(\varphi) \right) \right]}$$

Note: $\varphi(x_0 = 0) = \varphi_a(x_i)$ $\quad$ $\varphi(x_0 = T) = \varphi_b(x_i)$

Note exponent is quadratic in π.

$\underline{\pi = \pi' + \dot{\varphi}}$

$$= \int D\varphi\, D\pi' \exp\left[i \int_0^T d^4x \left(\tfrac{1}{2}\dot{\varphi}^2 - \tfrac{1}{2}\pi'^2 - \tfrac{1}{2}(\nabla\varphi)^2 - V(\varphi) \right) \right]$$

$$\propto \int D\varphi \exp\left[i \int_0^T d^4x \left(\tfrac{1}{2}(\partial_\mu\varphi)^2 - V(\varphi) \right) \right]$$

$\therefore$

$$\boxed{\langle \varphi_b(x_i) | e^{-iHT} | \varphi_a(x_i) \rangle = \int D\varphi \exp\left[i \int_0^T d^4x \, \mathcal{L} \right]}$$

constant in front

$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\varphi)^2 - V(\varphi)$ Lagrange density

Note:

- Manifestly Lorentz invariant (other than T dependence)
- Useful for consideration of symmetries (recall Noether)
- We will assume $\mathcal{L}$ to be most fundamental specification / definition of Quantum Field Theory

$$\boxed{\text{How do we get } \underline{\underline{\text{correlation functions}}} \text{ from the path integral?}}$$

Consider

$$\int \mathcal{D}\varphi(x)\ \varphi(x_1)\,\varphi(x_2)\,\exp\left[i\int_{-T}^{T} d^4x\ \mathcal{L}(\varphi)\right]$$

$\underline{\text{B.C.}}$

$$\boxed{\begin{aligned} \varphi(-T,x_i) &= \varphi_a(x) \\ \varphi(T,x_i) &= \varphi_b(x) \end{aligned}}$$

Break up measure:

$$\int \mathcal{D}\varphi(x) = \int \mathcal{D}\varphi_1(x) \int \mathcal{D}\varphi_2(x) \underbrace{\int_{\substack{\varphi(x_1^0,x_i)=\varphi_1(x) \\ \varphi(x_2^0,x_i)=\varphi_2(x)}} \mathcal{D}\varphi(x)}_{\text{constraint at intermediate step}}$$

$$\boxed{\begin{aligned} \varphi(x_1) &= \varphi_1(x_1) \\ \varphi(x_2) &= \varphi_2(x_2) \end{aligned}}$$

$\therefore$ we have

$$\int D\varphi(x)\ \varphi(x_1)\,\varphi(x_2)\ \exp\left[i\int_{-T}^{T} d^4x\ \mathcal{L}(\varphi)\right]$$

$$= \int D\varphi_1(x)\,D\varphi_2(x)\ \varphi_1(x_1)\,\varphi_2(x_2) \int D\varphi\ \exp\left[i\int_{-T}^{T} d^4x\ \mathcal{L}(\varphi)\right]$$

$$\left\{ \begin{array}{l} \varphi(-T) = \varphi_a \\ \varphi(x_1^0) = \varphi_1 \\ \varphi(x_2^0) = \varphi_2 \\ \varphi(T) = \varphi_b \end{array} \right\}$$

Now rewrite path integral as transition amplitude:

$$= \int D\varphi_1(x)\,D\varphi_2(x)\ \varphi_1(x_1)\,\varphi_2(x_2)$$

$$\times \langle \varphi_b | e^{-iH(T-x_2^0)} | \varphi_2 \rangle \langle \varphi_2 | e^{-iH(x_2^0-x_1^0)} | \varphi_1 \rangle$$

$$\times \langle \varphi_1 | e^{-iH(x_1^0+T)} | \varphi_a \rangle$$

Now turn φ_1, φ_2 into Schrödinger operators:

$$\boxed{\begin{array}{l} \varphi_1(x_1)\,|\varphi_1\rangle = \varphi_S(x_1^i)\,|\varphi_1\rangle^{t} \\ \varphi_2(x_2)\,|\varphi_2\rangle = \varphi_S(x_2^i)\,|\varphi_2\rangle^{t} \end{array}}$$

Can then use completeness:

$$\int D\varphi_1 \, |\varphi_1\rangle\langle\varphi_1| = \int D\varphi_2 \, |\varphi_2\rangle\langle\varphi_2| = \mathbb{1}$$

$$\int D\varphi(x) \, \varphi(x_1)\,\varphi(x_2) \exp\left[i \int_{-T}^{T} d^4x \, \mathcal{L}(\varphi) \right]$$

$$= \langle\varphi_b| e^{-iH(T-x_2^0)} \varphi_S(\vec{x}_2) e^{-iH(x_2^0-x_1^0)} \varphi_S(\vec{x}_1) e^{-iH(x_1^0+T)} |\varphi_a\rangle$$

{ Exponentials combine w/ φ_S to give φ_H !! }

For $x_1^0 > x_2^0$ order of x_1 and x_2 would be interchanged !!

$$= \langle\varphi_b| e^{-iHT} \, T\{\varphi_H(x_1)\varphi_H(x_2)\} \, e^{-iHT} |\varphi_a\rangle$$

Recall: Take limit $T \to \infty(1-i\epsilon)$ $\Rightarrow$

$$e^{-iHT}|\varphi_a\rangle \simeq e^{-iE_0\cdot\infty(1-i\epsilon)} \langle\Omega|\varphi_a\rangle |\Omega\rangle$$

We can get rid of constant junk by normalizing:

$$\boxed{\langle\Omega| T\{\varphi_H(x_1)\varphi_H(x_2)\}|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\int D\varphi \, \varphi(x_1)\varphi(x_2) \, e^{i\int_{-T}^{T} d^4x \, \mathcal{L}}}{\int D\varphi \, e^{i\int_{-T}^{T} d^4x \, \mathcal{L}}}}$$

Straightforward generalization to n-point functions !!

Generating Functional

Recall rules for differentiating discrete vectors:

$$\frac{\partial}{\partial x_i} x_j = \delta_{ij} \qquad \frac{\partial}{\partial x_i} \sum_j x_j k_j = \sum_j \delta_{ij} k_j = k_i$$

Functional Differentiation

$$\frac{\delta}{\delta J(x)} J(y) = \delta^{(4)}(x-y)$$

$$\frac{\delta}{\delta J(x)} \int d^4y\, J(y)\,\varphi(y) = \int d^4y\, \delta^{(4)}(x-y)\,\varphi(y) = \varphi(x)$$

Used rules of differentiation only.

Consider:

$$\frac{\delta}{\delta J(x)} \int d^4y\, \partial_\mu J(y)\, V^\mu(y)$$

$$= \frac{\delta}{\delta J(x)} \left(- \int d^4y\, J(y)\, \partial_\mu V^\mu(y) \right) = - \partial_\mu V^\mu(x)$$