Path Integrals in Quantum Mechanics

- Useful to see alternate derivation of Feynman Rules
- close analogy between Q.F.T. and STAT MECH.

Consider non-relativistic Q.M. in one dimension:

$$H = \frac{p^2}{2m} + V(x)$$

Suppose we wish to describe a particle travelling from position x_a to position x_b in time T.

Amplitude for this process is

$$\boxed{U(x_a, x_b; T) = \langle x_b | e^{-iHT/\hbar} | x_a \rangle}$$

(in "canonical" Hamiltonian formalism)

Let's motivate a different mathematical expression for this amplitude.

{ Superposition principle: amplitude is coherent sum of all ways in which process can take place. }

In our example, consider the amplitude for each particular spatial path that particle can take as a pure phase. Then expect:

(*) $$\boxed{U(x_a, x_b; T) = \sum_{\text{all paths}} e^{i\,\text{phase}} = \int \mathcal{D}x(t)\, e^{i\,\text{phase}}}$$

{ indicates 1 path for every function $x(t)$ that begins at x_a and ends at x_b }

Aside on Functionals

Functional: A function that maps functions to numbers.

{ e.g. $F[x(t)]$ ← note square brackets is a functional.

Can be integrated over a set of functions $x(t)$. }

We will see more properties of functionals as we proceed.

What do we use for "phase" in (*) ??

In classical limit there should be 1 path: the classical path

One possibility is that "stationary phase approximation" picks out classical path.

{ Recall: If $\underbrace{I(k) = \int dx\, g(x)\, e^{ikf(x)}}$

S.P.A. $\Rightarrow$ $I(k)$ depends only on critical points of $f(x)$: i.e. when $f'(x)\big|_{x=x_c} = 0$ }

Hence would expect classical path $x_{cl}(x)$ to be given by:

$$\boxed{\frac{\delta}{\delta x(t)}\left(\text{phase}[x(t)]\right)\Big|_{x=x_{cl}} = 0}$$

However, we also know that

$$\boxed{\frac{\delta}{\delta x(t)} \underbrace{S}_{\text{action}}[x(t)]\Big|_{x=x_{cl}} = 0}$$

"principle of least action"

$$S = \int dt\, L$$

$$\Rightarrow \boxed{\text{phase} = \frac{S}{\hbar}}$$

$\leftarrow$ constant w/ units of action

{ S.P.A. valid when $S \gg \hbar$ }

$\therefore$ we have the conjecture:

$$\boxed{\langle x_b | e^{-iHT} | x_a \rangle = U(x_a, x_b; T) = \int \mathcal{D}x(t)\, e^{iS[x(t)]/\hbar}}$$

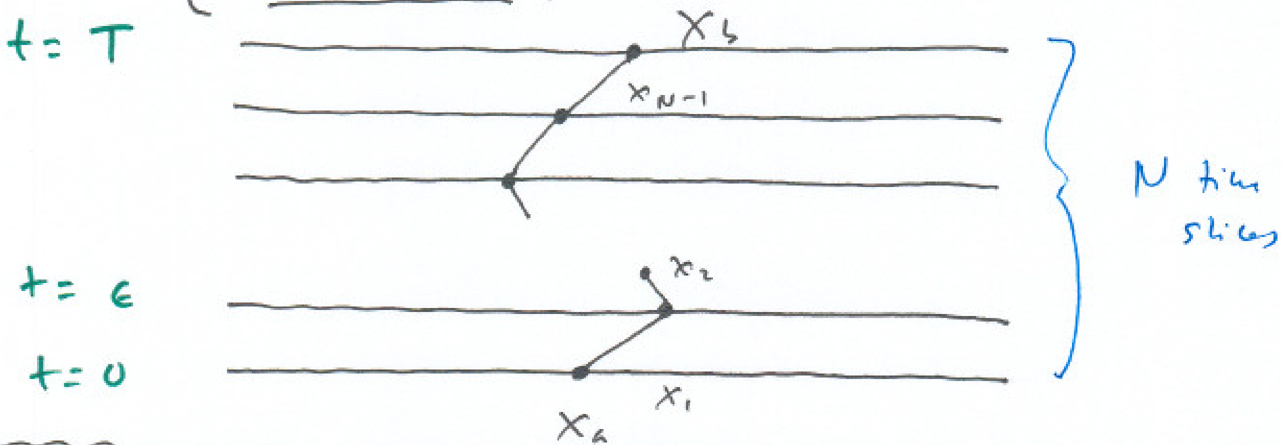
canonical Hamiltonian formulation of Q.M.

path integral formulation of Q.M.

To evaluate the path integral explicitly, we must define: $\int \mathcal{D}x(t)$

Use <u>discretization</u>: break up time interval 0 to T into many small pieces of duration ϵ.

[Approximate a "path" $x(t)$ as a <u>sequence of straight lines</u>, one for each time slice.]



<u>Discretized action</u>:

$$\boxed{S = \int_0^T dt \left(\frac{m}{2} \dot{x}^2 - V(x) \right) \rightarrow \sum_k \left(\frac{m}{2} \frac{(x_{k+1} - x_k)^2}{\epsilon} - \epsilon V\left(\frac{x_{k+1} + x_k}{2} \right) \right)}$$

Define the path integral measure by:

$$\boxed{\int \mathcal{D}x(t) \equiv \frac{1}{C(\epsilon)} \int \frac{dx_1}{C(\epsilon)} \int \frac{dx_2}{C(\epsilon)} \cdots \int \frac{dx_{N-1}}{C(\epsilon)} = \frac{1}{C(\epsilon)} \prod_k \int_{-\infty}^{\infty} \frac{dx_k}{C(\epsilon)}}$$

$C(\epsilon) =$ const

Reason for this definition will become clear below

Now we would like to show equivalence of "canonical" and "path integral" approaches for 1 particle potential problem by showing that both amplitudes satisfy the same differential equation, with same initial condition.

Consider the addition of the very last time slice in discretized sum over paths:

$$U(x_a, x_b; T) = \int_{-\infty}^{\infty} \frac{dx'}{C(\epsilon)} \exp\left[\frac{i}{\hbar} \frac{m(x_b - x')^2}{2\epsilon} - \frac{i}{\hbar} \epsilon V\left(\frac{x_b + x'}{2}\right)\right] U(x_a, x'; T-\epsilon)$$

$\Downarrow$ $U(x'; x_b; \epsilon)$

Now consider limit $\epsilon \to 0$ oscillates rapidly!!

constrains x' to be near x_b! $\Rightarrow$ EXPAND!

Expand in powers of $\delta \equiv (x' - x_b)$:

$$\left\{ \begin{aligned} V\left(\frac{x_b + x'}{2}\right) &= V(x_b + \delta/2) \simeq V(x_b) \\ U(x_a, x'; T-\epsilon) &= U(x_a, x_b + \delta; T-\epsilon) \\ &\simeq \left[1 + \delta \frac{\partial}{\partial x_b} + \frac{1}{2}\delta^2 \frac{\partial^2}{\partial x_b^2} + \dots\right] U(x_a, x_b; T-\epsilon) \end{aligned} \right\}$$

$$\boxed{\begin{aligned} U(x_a, x_b; T) = \int_{-\infty}^{\infty} \frac{dx'}{C} \exp\left(\frac{i}{\hbar}\frac{m}{2\epsilon}(x_b - x')^2\right)\left[1 - \frac{i\epsilon}{\hbar} V(x_b)\right] \\ \times \left[1 + (x' - x_b)\frac{\partial}{\partial x_b} + \frac{1}{2}(x' - x_b)^2 \frac{\partial^2}{\partial x_b^2} + \dots\right] U(x_a, x_b; T-\epsilon) \end{aligned}}$$

We can now do the "Gaussian" integrals over x'!!

$$\left\{ \int dy\, e^{-by^2} = \sqrt{\frac{\pi}{b}} \qquad \int dy\, y\, e^{-by^2} = 0 \qquad \int dy\, y^2 e^{-by^2} = \frac{1}{2b}\sqrt{\frac{\pi}{b}} \right\}$$

$\Rightarrow$

$$\boxed{\begin{aligned} U(x_a, x_b; T) = \left(\frac{1}{C}\sqrt{\frac{2\pi\hbar\epsilon}{-im}}\right)\left[1 - \frac{i\epsilon}{\hbar}V(x_b) + \frac{i\epsilon\hbar}{2m}\frac{\partial^2}{\partial x_b^2} + O(\epsilon^2)\right] \\ \times U(x_a, x_b; T-\epsilon) \end{aligned}}$$

As $\epsilon \to 0$ makes sense only if $(\;) = 1$

$$\Rightarrow \boxed{C(\epsilon) = \sqrt{\frac{2\pi\hbar\epsilon}{-im}}}$$

Now we can further expand $U(x_a, x_b; T-\epsilon)$:

$$U(x_a, x_b; T) = (1)\left[1 - \frac{i\epsilon}{\hbar} V(x_b) + \frac{i\epsilon\hbar}{2m}\frac{\partial^2}{\partial x_b^2} + O(\epsilon^2)\right]$$

$$\times\left[1 - \epsilon\frac{\partial}{\partial T} + O(\epsilon^2)\right] U(x_a, x_b; T)$$

Match at $O(\epsilon)$:

$$\boxed{\begin{aligned} i\hbar\frac{\partial}{\partial T} U(x_a, x_b; T) &= \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_b^2} + V(x_b)\right] U(x_a, x_b; T) \\ &= H\, U(x_a, x_b; T) \end{aligned}}$$

Schrödinger Equation !!

{ of course $U(x_a, x_b; T)$ in canonical formalism also satisfies this equation !! }

What about initial condition ??

canonical: As $T \to 0$ $\quad \langle x_b | e^{-iHT/\hbar} | x_a \rangle \to \delta(x_a - x_b)$

path integral: As $T \to 0$ $\quad \frac{1}{C(\epsilon)} \exp\left[\frac{i}{\hbar}\frac{m(x_b - x_a)^2}{2\epsilon}\right] \to \delta(x_a - x_b)$

$\Rightarrow$ SAME INITIAL CONDITION !! ✓ $\quad$ 1 time slice!