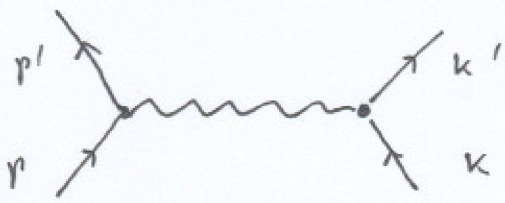
Simplest application of Q.E.D
Feynman Rules:

Coulomb Potential

(Recall Yukawa potential)

$i\mathcal{M} =$ 

$(ff \to ff)$

$$= (-ie)^2\, \bar{u}(p')\gamma^\mu u(p)\, \frac{-i g_{\mu\nu}}{(p'-p)^2}\, \bar{u}(k')\gamma^\nu u(k)$$

Non-relativistic Limit ??

Again: $\quad \bar{u}(p')\gamma^0 u(p) = u^\dagger(p')u(p) \simeq +2m\,\xi'^\dagger\xi$

$$\text{So} \quad i\mathcal{M} \simeq \frac{+ie^2}{-|p_i'-p_i|^2}\,(2m\,\xi'^\dagger\xi)_p\,(2m\,\xi'^\dagger\xi)_k \cdot g_{00}$$

$$= \frac{-ie^2}{|p_i'-p_i|^2}\,(2m\,\xi'^\dagger\xi)(2m\,\xi'^\dagger\xi)$$

minus as compared w/ Yukawa!!

Match to N.R. Q.M.

Coulomb Potential

$$\boxed{V(r) = \frac{e^2}{4\pi r} = \frac{\alpha}{r}}$$

$\left\{ \alpha = \frac{e^2}{4\pi} \sim \frac{1}{137} \right\}$

Fine structure constant

Repulsive interaction !!

Other potentials ??

Exchange particle	ff $(\bar{f}\bar{f})$	$f\bar{f}$
Scalar (Yukawa)	A	A
Vector (QED)	R	A
Tensor (gravity)	A	A

Let's consider important Q.E.D. process and go through full calculation:

$$\boxed{e^+e^- \to \mu^+\mu^-}$$

Fundamental importance in high-energy physics: Calibration for e^+e^- colliders.

$e^+e^- \to \bar{q}q$ important probe of Standard Model

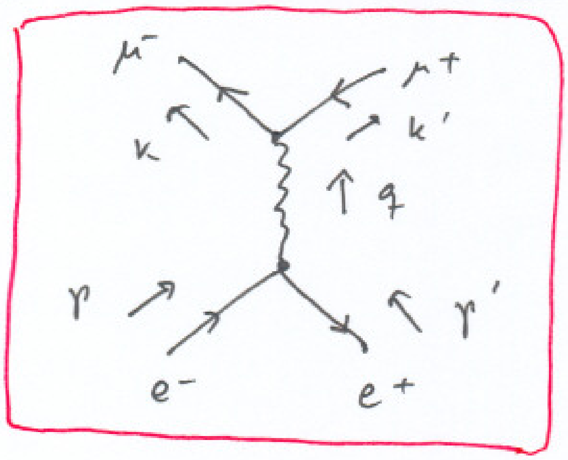
We will compute:

Unpolarized cross-section for $e^+e^- \to \mu^+\mu^-$

at lowest order in perturbation theory.

Assume: $m_e = 0$ $\quad$ $m_\mu \neq 0$

$$\left\{ \frac{m_e}{m_\mu} \sim \frac{1}{200} \ll 1 \right\}$$

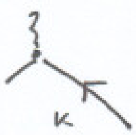


Leading-order diagram

(arrows denote direction of charge flow)

Recall Feynman Rules

 $= \bar{v}^s(k)$ $\quad$ $\left(\sim \bar{\psi}\,|k:, s\rangle\right)$

 $= u^s(p)$ $\quad$ $\left(\sim \psi\,|k:, s\rangle\right)$

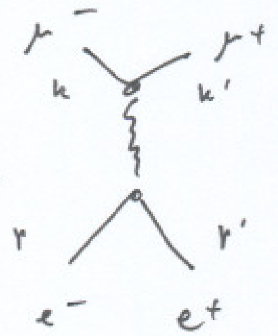
$= -ie\gamma^\mu$

 $= \bar{u}^s(k)$ $\quad$ $\left(\sim \langle k:, s|\,\bar{\psi}\right)$

$= \dfrac{-ig_{\mu\nu}}{q^2+i\epsilon}$

 $= v^s(k)$ $\quad$ $\left(\sim \langle k:, s|\,\psi\right)$

$i\mathcal{M} =$



$$= \bar{v}^{s'}(p')(-ie\gamma^\mu)u^s(p)\left(\frac{-ig_{\mu\nu}}{q^2+i\epsilon}\right)\bar{u}^r(k)(-ie\gamma^\nu)v^{r'}(k')$$

$$\Rightarrow \boxed{i\mathcal{M}\left(e^-(p)e^+(p') \to \mu^-(k)\mu^+(k')\right) = \frac{ie^2}{q^2}\left(\bar{v}(p')\gamma^\mu u(p)\right)\left(\bar{u}(k)\gamma_\mu v(k')\right)}$$

We need $|\mathcal{M}|^2$ to get cross-section.

Need complex conjugate bilinears:

$$\left\{ \begin{aligned} (\bar{v}\gamma^\mu u)^* &= (\bar{v}\gamma^\mu u)^\dagger = u^\dagger(\gamma^\mu)^\dagger(\gamma^0)^\dagger v \\ &= u^\dagger(\gamma^\mu)^\dagger\gamma^0 v = u^\dagger\gamma^0\gamma^0(\gamma^\mu)^\dagger\gamma^0 v \\ &= \bar{u}\gamma^\mu v \qquad (\text{as } \gamma^0\gamma^\mu\gamma^0 = \gamma^{\mu\dagger}) \end{aligned} \right\}$$

Thus,

$$|\mathcal{M}|^2 = \frac{e^4}{q^4}\left(\bar{v}^{s'}(p')\gamma^\mu \overset{1}{u^s(p)}\,\bar{u}^s(p)\gamma^\nu v^{s'}(p')\right) \times \left(\bar{u}^r(k)\gamma_\mu \overset{2}{v^{r'}(k')}\,\bar{v}^{r'}(k')\gamma^\nu u^r(k)\right)$$

Note: we are still free to specify the polarizations

(generally no summation convention intended!)

Most experiments are unpolarized:

- Average initial electron, positron spins
- Then, sum over final muon spins.

This gives big simplification:

$$\underbrace{\frac{1}{2}\sum_{s}\frac{1}{2}\sum_{s'}}_{\text{Average}}\ \underbrace{\sum_{r}\sum_{r'}}_{\text{Sum}}\ \left|\mathcal{M}(s s' \to r r')\right|^2$$

We can do spin sums using completeness relations found previously:

$$\boxed{\begin{aligned}\sum_s u^s(p)\,\bar{u}^s(p) &= \not{p} + m \\ \sum_s v^s(p)\,\bar{v}^s(p) &= \not{p} - m\end{aligned}}$$

Let's write out ① with all indices explicit:

$$\sum_{s,s'} ① = \sum_{s,s'} \bar{v}^{s'}_a(p')\,\gamma^\mu_{ab}\,u^s_b(p)\,\bar{u}^s_c(p)\,\gamma^\nu_{cd}\,v^{s'}_d(p')$$

$$= \sum_{s,s'} \underbrace{v^{s'}_d(p')\,\bar{v}^{s'}_a(p')}\,\gamma^\mu_{ab}\,\underbrace{u^s_b(p)\,\bar{u}^s_c(p)}\,\gamma^\nu_{cd}$$

$$= \underbrace{(\not{p}' - m)_{da}\,\gamma^\mu_{ab}\,(\not{p} + m)_{bc}\,\gamma^\nu_{cd}}$$

indicates Trace!

$$\sum_{s,s'} ① = \mathrm{Tr}\left((\not{p}' - m)\gamma^\mu (\not{p} + m)\gamma^\nu \right)$$

and similarly

$$\sum_{r,r'} ② = \mathrm{Tr}\left((\not{k} + m)\gamma_\mu (\not{k}' - m)\gamma_\nu \right)$$

Putting it together:

$$\boxed{\frac{1}{4}\sum_{\text{spins}} |M|^2 = \frac{e^4}{4q^4}\, \mathrm{Tr}\left[(\not{p}' - m_e)\gamma^\mu (\not{p} + m_e)\gamma^\nu \right] \times \mathrm{Tr}\left[(\not{k} + m_\mu)\gamma_\mu (\not{k}' - m_\mu)\gamma_\nu \right]}$$

We've reduced $\sum |M|^2$ to traces over γ matrices!!

Interlude

Recall that γ's form a group theoretical structure: an algebra

$$\underline{\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\,\mathbb{1}} \qquad \left(\text{e.g. } S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \right)$$

We need some new identities.

Recall:
$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \quad \left\{ \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \right\}$$

$$\underline{\mathrm{Tr}\,\mathbb{1} = 4} \quad \text{in spinor space.}$$

$\mathrm{Tr}\,\gamma^\mu = 0$ (by inspection)

More systematic:

$\{\gamma^\mu, \gamma^5\} = 0$

$$\mathrm{Tr}\,\gamma^\mu = \mathrm{Tr}(\gamma^5\gamma^5\gamma^\mu) = -\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^5)$$

$$= -\mathrm{Tr}(\gamma^5\gamma^5\gamma^\mu) = -\mathrm{Tr}\,\gamma^\mu$$

cyclic property

$\Rightarrow$ $\mathrm{Tr}\,\gamma^\mu = 0$

For product of n gamma matrices get a minus from anticommutation $\Rightarrow$

$$\boxed{\mathrm{Tr}(\text{odd } \# \ \gamma\text{'s}) = 0}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = \mathrm{Tr}(2g^{\mu\nu}\mathbb{1} - \gamma^\nu\gamma^\mu)$$

$$= 8g^{\mu\nu} - \mathrm{Tr}(\gamma^\mu\gamma^\nu)$$

cyclic

$$\Rightarrow \boxed{\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}}$$

{ Can do similar procedure for arbitrary even # of γ's but gets complicated quickly }

Let's do 4 γ's:

$$\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = \mathrm{tr}\left[(2g^{\mu\nu} \mathbb{1} - \gamma^\nu \gamma^\mu) \gamma^\rho \gamma^\sigma \right]$$

$$= \mathrm{tr}\left[2g^{\mu\nu} \gamma^\rho \gamma^\sigma - \gamma^\nu (2g^{\mu\rho} \mathbb{1} - \gamma^\rho \gamma^\mu) \gamma^\sigma \right]$$

$$= \mathrm{tr}\left[2g^{\mu\nu} \gamma^\rho \gamma^\sigma - 2g^{\mu\rho} \gamma^\nu \gamma^\sigma + \gamma^\nu \gamma^\rho \gamma^\mu \gamma^\sigma \right]$$

$$= \mathrm{tr}\left[2g^{\mu\nu} \gamma^\rho \gamma^\sigma - 2g^{\mu\rho} \gamma^\nu \gamma^\sigma + 2g^{\mu\sigma} \gamma^\nu \gamma^\rho - \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\mu \right]$$

$$= 2g^{\mu\nu} \mathrm{Tr}(\gamma^\rho \gamma^\sigma) - 2g^{\mu\rho} \mathrm{Tr}(\gamma^\nu \gamma^\sigma) + 2g^{\mu\sigma} \mathrm{Tr}(\gamma^\nu \gamma^\rho)$$
$$- \mathrm{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma)$$

$$\Rightarrow \boxed{\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4\left(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho} \right)}$$

{ can always reduce trace of n γ's to
<u>sum</u> of traces of $n-2$ γ's }

Recall: $\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3$

← even # of γ's

$$\boxed{\mathrm{Tr}\left(\gamma^5 (\text{odd \# of } \gamma\text{'s})\right) = 0}$$

$$\mathrm{Tr}\,\gamma^5 = \mathrm{tr}(\gamma^0 \gamma^0 \gamma^5) = -\mathrm{Tr}(\gamma^0 \gamma^5 \gamma^0) = -\mathrm{Tr}(\gamma^0 \gamma^0 \gamma^5)$$
$$= -\mathrm{Tr}\,\gamma^5$$

$$\boxed{\mathrm{Tr}\,\gamma^5 = 0}$$

Similarly one can show

$$\boxed{Tr(\gamma^\mu\gamma^\nu\gamma^5) = 0}$$

$$\boxed{Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = -4i\,\epsilon^{\mu\nu\rho\sigma}}$$

There are also useful "contraction" identities, some of which you proved in homework.

e.g. $\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu$ etc.

{ Note: $\gamma^\mu\gamma_\mu = g_{\mu\nu}\gamma^\mu\gamma^\nu = \frac{1}{2} g_{\mu\nu}\{\gamma^\mu,\gamma^\nu\}$
$= g_{\mu\nu}g^{\mu\nu} = 4$ }

Now let's return to $e^+e^- \to \mu^+\mu^-$ (unpolarized)

$$Tr[(\not{p}' - m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu]$$

$$= Tr[\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu p'_\rho p_\sigma - m_e^2\gamma^\mu\gamma^\nu]$$

+ (cross terms w/ odd # of γ's) $\to 0$

$$= p'_\rho p_\sigma\, 4(g^{\rho\mu}g^{\sigma\nu} - g^{\rho\sigma}g^{\mu\nu} + g^{\rho\nu}g^{\mu\sigma})$$

$$- m_e^2\, 4 g^{\mu\nu}$$

$$= 4\left(p'^\mu p^\nu - p'\cdot p\, g^{\mu\nu} + p'^\nu p^\mu\right) - m_e^2\, 4\, g^{\mu\nu}$$

$\Rightarrow$

$$\boxed{\mathrm{Tr}\left[(\not p' - m_e)\gamma^\mu(\not p + m_e)\gamma^\nu\right] = 4\left[p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu} p'\cdot p\right]}$$

(neglected m_e^2 term!)

Similarly,

$$\boxed{\mathrm{Tr}\left[(\not k + m_\mu)\gamma_\mu(\not k' - m_\mu)\gamma_\nu\right] = 4\left[k_\mu k_\nu' + k_\nu k_\mu' - g_{\mu\nu}(k\cdot k' + m_\mu^2)\right]}$$

$\Downarrow$

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{4e^4}{q^4}\left(p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu} p'\cdot p\right) \times \left[k_\mu k_\nu' + k_\nu k_\mu' - g_{\mu\nu}(k\cdot k' + m_\mu^2)\right]$$

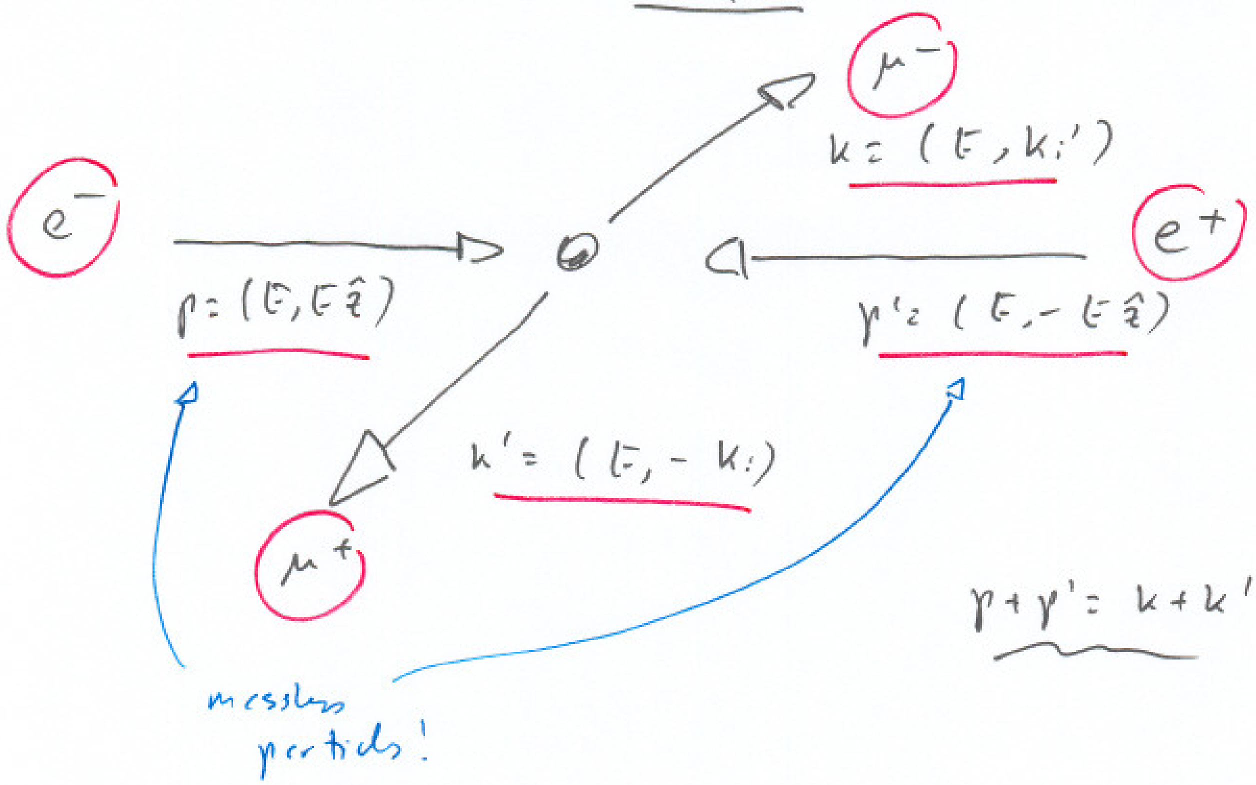
$$\boxed{\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{8e^4}{q^4}\left(p'\cdot k\; p\cdot k' + p'\cdot k'\, p\cdot k + m_\mu^2\, p\cdot p'\right)}$$

(fully Lorentz invariant!)

Recall general procedure:

- Draw diagrams ✓
- Obtain $\mathcal{M}$ from F.R. ✓
- $|\mathcal{M}|^2$ + deal w/ polarization ✓ $\Leftarrow$ Totally generic procedure!!
- Evaluate traces ✓
- Go to useful Reference Frame
- Plug $|\mathcal{M}|^2$ into σ + do phase space

Let's work in Center of Mass Frame:



$p + p' = k + k'$

$k_i \hat{z}_i = |k_i| \cos\theta \quad , \quad |k_i| = \sqrt{E^2 - m_\mu^2}$

$q \equiv p + p'$

$$q^2 = p^2 + p'^2 + 2\, p\cdot p' = 0 + 0 + 4E^2 \qquad p\cdot p' = 2E^2$$

$$\left\{ \begin{aligned} p\cdot k &= E^2 - E|k_i|\cos\theta = p'\cdot k' \\ &= E(E - |k_i|\cos\theta) \\ p\cdot k' &= E(E + |k_i|\cos\theta) = p'\cdot k \end{aligned} \right\}$$

Now take and plug into:

$$\frac{1}{4}\sum_{\text{spins}} |M|^2 = \frac{8e^4}{q^4}\left(p'\cdot k\; p\cdot k' + p'\cdot k'\; p\cdot k + m_\mu^2\, p\cdot p'\right)$$

$\Rightarrow$

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{8e^4}{16E^4}\Big(E^2(E+|k_i|\cos\theta)^2 + E^2(E-|k_i|\cos\theta)^2 + 2m_\mu^2 E^2\Big)$$

$$= \frac{e^4}{2E^4}\left(2E^4 + 2E^2|k_i|^2\cos^2\theta + 2m_\mu^2E^2\right)$$

$$= e^4\left(1 + \frac{|k_i|^2}{E^2}\cos^2\theta + \frac{m_\mu^2}{E^2}\right)$$

$$= e^4\left(1 + \frac{(E^2-m_\mu^2)}{E^2}\cos^2\theta + \frac{m_\mu^2}{E^2}\right)$$

$$\Rightarrow \boxed{\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = e^4\left[\left(1+\frac{m_\mu^2}{E^2}\right) + \left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]}$$

We are ready for the last step!!

Recall cross-section formula:

$$\left(\frac{d\sigma}{d\Omega}\right)_{c.m.} = \frac{1}{2E_AE_B|v_A-v_B|}\frac{|\vec{p}_i|}{(2\pi)^2 4E_{c.m.}}|\mathcal{M}(p_Ap_B\to p_1p_2)|^2$$

$$|v_A-v_B| = \left|\frac{k_A^z}{E_A} - \frac{k_B^z}{E_B}\right| = \left|\frac{E}{E} - \left(-\frac{E}{E}\right)\right| = 2$$

$$E_A = E_B = E = \frac{E_{c.m.}}{2} \qquad 2E_AE_B = 2\left(\frac{E_{c.m.}}{2}\right)\left(\frac{E_{c.m.}}{2}\right) = \frac{E_{c.m.}^2}{2}$$

Plug everything in $\Rightarrow$

$$\left(\frac{d\sigma}{d\Omega}\right)_{c.m.} = \frac{1}{\frac{E_{c.m.}^2}{2}\, 2} \frac{|k_i|}{(2\pi)^2\, 4\, E_{c.m.}}\, e^4 \left[\left(1+\frac{m_\mu^2}{E^2}\right) + \left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]$$

$$= \frac{|k_i|}{E_{c.m}^3}\, \alpha^2 \left[\left(1+\frac{m_\mu^2}{E^2}\right) + \left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]$$

$$= \frac{\sqrt{E^2-m_\mu^2}}{E_{c.m}^2\, 2E}\, \alpha^2 [\;"\;]$$

$$\Rightarrow \boxed{\left(\frac{d\sigma}{d\Omega}\right)_{c.m.} = \frac{\alpha^2}{2E_{c.m.}^2} \sqrt{1-\frac{m_\mu^2}{E^2}} \left[\left(1+\frac{m_\mu^2}{E^2}\right) + \left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]}$$

"phase space" prefactor

Now let's integrate differential cross-section to get total cross section.

$$\int d\Omega = 2\pi \int_{-1}^{1} d\cos\theta$$

$$\left\{ \int_{-1}^{1} d\cos\theta = 2 \qquad \int_{-1}^{1} d\cos\theta\, \cos^2\theta = \frac{2}{3} \right\}$$

$$\sigma_{tot} = \frac{4\pi}{2E_{c.m}^2}\, \alpha^2 \sqrt{1-\frac{m_\mu^2}{E^2}} \left[\left(1+\frac{m_\mu^2}{E^2}\right) + \frac{1}{3}\left(1-\frac{m_\mu^2}{E^2}\right)\right]$$

$$\Rightarrow \boxed{\sigma_{tot} = \frac{8\pi\alpha^2}{3E_{c.m.}^2} \sqrt{1-\frac{m_\mu^2}{E^2}} \left[1+\frac{1}{2}\frac{m_\mu^2}{E^2}\right]}$$

Consider high-energy limit:

$$E \gg m_\mu$$

$$\sigma_{tot} \to \frac{8\pi}{3}\frac{\alpha^2}{E_{c.m.}^2} + \cdots$$

As $E_{c.m.} \to \infty$, $E_{c.m.}$ is only scale!

From dimensional analysis one would guess:

$$\sigma_{tot} \sim \frac{\alpha^2}{E_{c.m.}^2}$$

So all the work we've done in this course so far has been to calculate $\frac{8\pi}{3}$!!

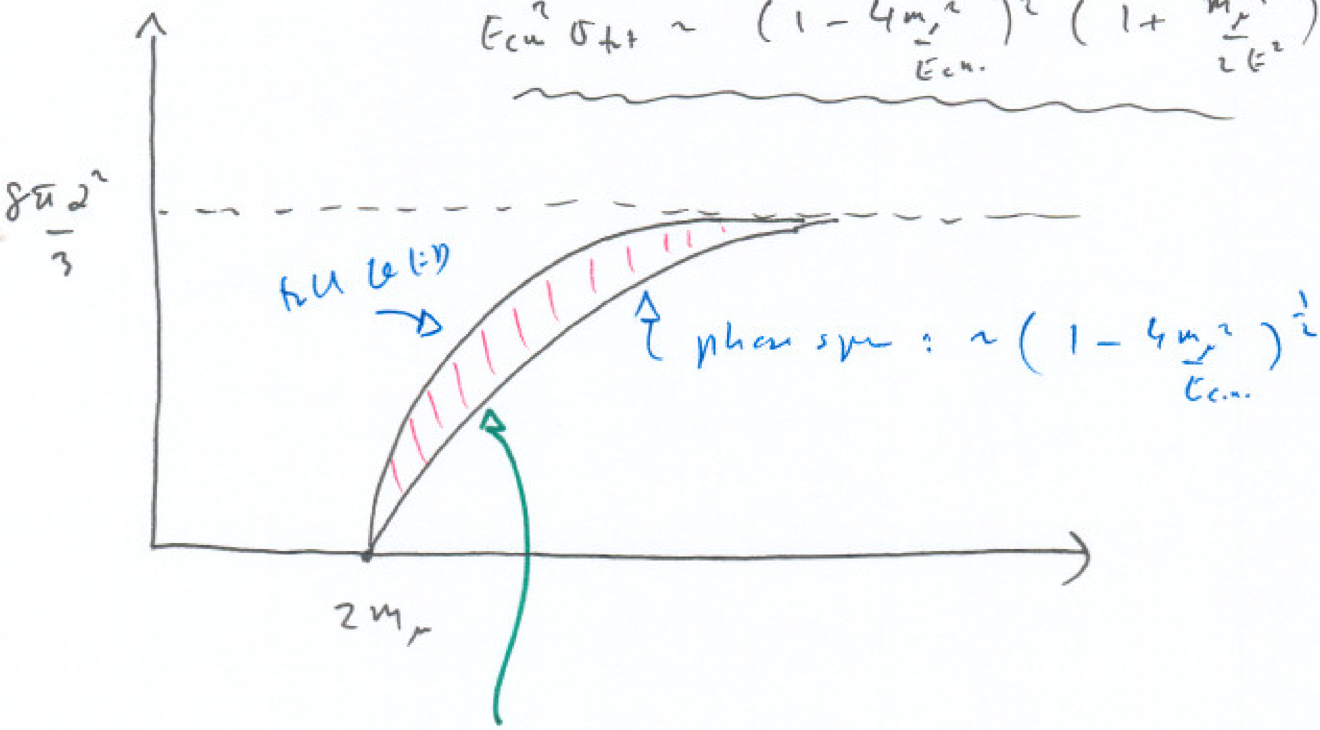
Consider the energy dependence in more detail.

$$\sigma_{tot} \sim \underbrace{\frac{1}{E_{cm}^2}\left(1-\frac{m_\mu^2}{E^2}\right)^{\frac{1}{2}}}_{\text{phase space}} \quad \underbrace{\left(1+\frac{1}{2}\frac{m_\mu^2}{E^2}\right)}_{\text{QED dynamics}}$$

How do we test QED using this formula??

Plot: $E_{cm}^2 \sigma_{tot}$ vs. E_{cm}

$$E_{cm}^2 \sigma_{tot} \sim \left(1 - \frac{4m_\mu^2}{E_{cm}}\right)^{\frac{1}{2}} \left(1 + \frac{m_\mu^2}{2E^2}\right)$$



To test QED must be able to resolve the difference from naive phase space prediction.

Experimental production of $\mu^+\mu^-$ and $\tau^+\tau^-$ confirms QED prediction for LO σ_{tot}!!

One can use formula to fit τ mass!!

$$\Rightarrow \quad m_\tau = 1782^{+2}_{-7} \text{ MeV}$$

$$\left(m_\tau^{\text{today}} = 1776.99 \pm 0.29 \text{ MeV}\right)$$