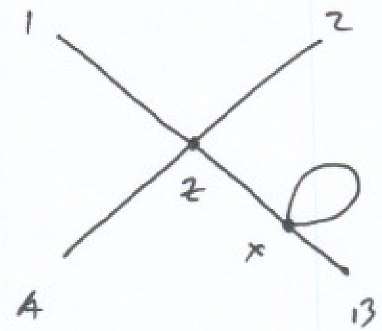
Let's consider  in some detail.

Arises from contraction:

$$\langle p_1 p_2 |\, \tfrac{1}{2}\left(\frac{-i\lambda}{4!}\int d^4x\, \varphi\varphi\varphi\varphi\right)\left(\frac{-i\lambda}{4!}\int d^4z\, \varphi\varphi\varphi\varphi\right) | p_A\, p_B \rangle$$



Using coordinate-space Feynman Rules $\Rightarrow$

$$(-i\lambda)^2 \int d^4x\, D(x-x)\, e^{-ip_B\cdot x} \int d^4z\, D(x-z)\, e^{i(p_1+p_2-p_A)\cdot z}$$

$$\left\{ \text{Recall: } D(x-z) = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2}\, e^{-ip\cdot(x-z)} \right\}$$

$$= (-i\lambda)^2 \int d^4x \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2}\, e^{-ip_B\cdot x} \int d^4z \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}\, e^{-i(x-z)k} \times e^{i(p_1+p_2-p_A)\cdot z}$$

$$= (-i\lambda)^2 \underbrace{\int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2}}\ \underbrace{\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}}\ \underbrace{\int d^4x\, e^{-i(p_B+k)\cdot x}}_{\delta\text{-fn}}\ \underbrace{\int d^4z\, e^{i(k+p_1+p_2-p_A)\cdot z}}_{\delta\text{-fn}}$$

 $= (-i\lambda)^2 \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2-m^2} \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2-m^2} (2\pi)^4 \, \delta^{(4)}(p_B+k)$

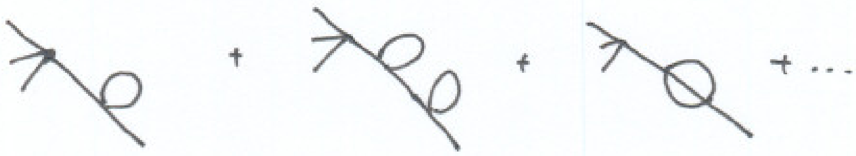
$\quad \delta^{(4)}(k+p_1+p_2-p_A)$

Now integrate over k using

$$\Rightarrow \left. \frac{i}{k^2-m^2} \right|_{k^2=p_B^2} = \frac{i}{p_B^2-m^2} = \frac{1}{0} \quad \text{as } p_B^2 = m^2$$

"on shell"
External
particle

But notice that contributions of form:



have nothing to do with scattering.

These diagrams "dress" the external state:

$$\boxed{|p\rangle_0 \longrightarrow |p\rangle}$$

We want to get rid of these diagrams as got rid of disconnecteds.

> **AMPUTATION:** Starting from external leg, find last point at which diagram can be cut by removing single propagator such that this operation separates the leg from rest of diagram.

E.g.

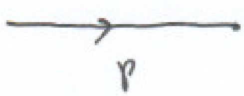
SUMMARY:

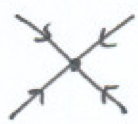
$$\boxed{\begin{array}{l} i\mathcal{M}\cdot(2\pi)^4\,\delta^{(4)}(p_A+p_B-\Sigma p_f) = \\ \text{Sum of all connected, amputated Feynman diagrams} \\ \text{w/ } p_A, p_B \text{ incoming and } p_f \text{ outgoing} \end{array}}$$

Momentum space Feynman Rules for $\mathcal{M}$:

$i\mathcal{M}$ = sum of all connected, amputated diagrams

where:

 $\frac{i}{p^2-m^2+i\epsilon}$

 $= -i\lambda$

Integrate $\int \frac{d^4p}{(2\pi)^4}$ over loop momenta

Divide by S

Fermions + Feynman Rules

Let's generalize definitions of time-ordering and normal ordering.

Recall:

$$T\{\varphi(x)\,\varphi(y)\} = \begin{cases} \varphi(x)\,\varphi(y) & x^0 > y^0 \\ \varphi(y)\,\varphi(x) & x^0 < y^0 \end{cases}$$

$$T\{\psi(x)\,\bar\psi(y)\} = \begin{cases} \psi(x)\,\bar\psi(y) & x^0 > y^0 \\ -\bar\psi(y)\,\psi(x) & x^0 < y^0 \end{cases}$$

Feynman propagator

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\, \frac{i(\not p + m)}{p^2 - m^2}\, e^{-ip\cdot(x-y)} = \langle 0|T\{\psi(x)\bar\psi(y)\}|0\rangle$$

What about time-ordering for product of more than 2 spinor fields??

E.g. $T\{\psi_1\psi_2\psi_3\psi_4\} = (-1)^3\,\psi_3\psi_1\psi_4\psi_2$

if $x_3^0 > x_1^0 > x_4^0 > x_2^0$

{ (−1) for each interchange }

Same for normal-ordering:

E.g. $N(a_p a_q a_r^\dagger) = (-1)^2 a_r^\dagger a_p a_q = (-1)^3 a_r^\dagger a_q a_p$

2 ways

Generalization of Wick's Theorem!

$$T\left(\psi(x)\bar{\psi}(y)\right) = N\left(\psi(x)\bar{\psi}(y)\right) + \overbrace{\psi(x)\bar{\psi}(y)}$$

WITH
$$\boxed{\begin{aligned} &\overbrace{\psi(x)\bar{\psi}(y)} = S_F(x-y) \\ &\overbrace{\psi(x)\psi(y)} = \overbrace{\bar{\psi}(x)\bar{\psi}(y)} = 0 \end{aligned}}$$

What about (−) signs ??

Define $N(\ )$ operation to account for (−) signs.

E.g. $$N(\overbrace{\psi_1 \psi_2 \bar{\psi}_3} \bar{\psi}_4) = -\overbrace{\psi_1 \bar{\psi}_3}\, N(\psi_2 \bar{\psi}_4)$$
$$= -S_F(x_1 - x_3)\, N(\psi_2 \bar{\psi}_4)$$

With these conventions Wick's Theorem takes same form as before:

$$\boxed{T(\psi_1 \bar{\psi}_2 \psi_3 \ldots) = N\left(\psi_1 \bar{\psi}_2 \psi_3 \ldots + \text{ALL POSSIBLE CONTRACTIONS}\right)}$$

(only minus sign different from bosonic case)

We are now ready to look at interesting examples of interacting field theories!!

(Feynman Rules for Fermions!!)

Yukawa Theory

Yukawa Interactions are very important in the Standard Model.

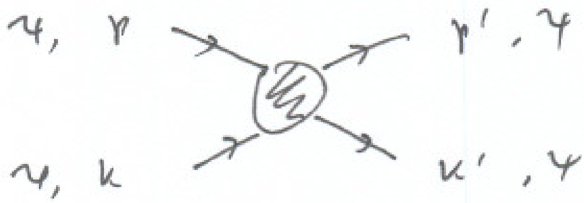
Recall: $H = H_0^{\text{Dirac}} + H_0^{KG} + \underline{\int d^3x \, g \bar{\psi} \psi \phi}$

(g )

Various types of 2-body scattering are possible:

$\underline{\psi\psi \to \psi\psi}$, $\bar{\psi}\bar{\psi} \to \bar{\psi}\bar{\psi}$, $\psi\bar{\psi} \to \phi\phi$, $\bar{\psi}\psi \to \bar{\psi}\psi$, $\psi\phi \to \psi\phi$

Consider:



Recall formula for T matrix:

$$\langle p_1 \cdots p_n | iT | p_A p_B \rangle = \lim_{T \to \infty(1-i\epsilon)} {}_0\langle p_1 \cdots p_n | T\left(e^{-i\int_{-T}^{T} H_I dt} \right) | p_A p_B \rangle_0$$

Here: $\underset{\sim\sim\sim}{H_I = g \bar{\psi}_I \psi_I \phi_I}$ Need H_I^2 !!

Leading contribution to scattering:

$$ {}_0\langle p' k' | T\left\{ \frac{1}{2!}(-ig)^2 \int d^4x\, \bar\psi_I \psi_I \varphi_I \int d^4y\, \bar\psi_I \psi_I \varphi_I \right\} | p k \rangle_0 $$

Note: $\mathcal{O}(g^2)$

Recall steps:

1. Use Wick's Thm to reduce T product to N product of contractions.
2. Act with uncontracted fields on external states.

First, need a little more technology:

$$ \psi_I(x)\, |p_i, s\rangle = \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{p'}}} \sum_{s'} a_{p_i'}^{s'} u^{s'}(p') e^{-ip'\cdot x} \sqrt{2E_{p_i}}\, a_{p_i}^{s\dagger} |0\rangle $$

$$ \left\{ a_{p'}^{s'} a_p^{s\dagger} \to (2\pi)^3 \delta^{(3)}(p_i - p_i')\, \delta^{s,s'} \right\} $$

$$ \Rightarrow \boxed{\begin{aligned} \psi_I(x)\, |p_i, s\rangle &= e^{-ip\cdot x} u^s(p)\, |0\rangle \\ \langle p_i s |\, \bar\psi_I(x) &= \langle 0 |\, e^{ip\cdot x} \bar u^s(p) \end{aligned}} $$

Same for antifermions:

$$ \boxed{\begin{aligned} \bar\psi_I(x)\, |p_i, s\rangle &= e^{-ip\cdot x} \bar v^s(p)\, |0\rangle \\ \langle p_i, s |\, \psi_I(x) &= \langle 0 |\, e^{ip\cdot x} v^s(p) \end{aligned}} $$

Schematically:

$$\langle p' k' |\ \bar\psi \psi \phi \ \bar\psi \psi \phi \ | p k \rangle$$

For $\psi\psi \to \psi\psi$ scattering, both $\bar\psi$'s must contract with final state momenta, both ψ's must contract with initial state momenta and the ϕ's must contract with each other.

typical contraction:

$$\langle p' k' | \frac{1}{2!} (-ig)^2 \int d^4x \ \bar\psi \psi \phi \int d^4y \ \bar\psi \psi \phi \ | p k \rangle$$

$$= (-ig)^2 \int d^4x \int d^4y \ e^{ik'\cdot x} \bar u(k') e^{-ik\cdot x} u(k) \, D_F(x-y) \, e^{ip'\cdot y} \bar u(p') e^{-ip\cdot y} u(p)$$

$$D_F(x-y) = \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m_\phi^2 + i\epsilon} e^{-iq\cdot(x-y)}$$

$$= (-ig)^2 \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m_\phi^2 + i\epsilon} \int d^4x \ e^{ix\cdot(k' - q - k)} \int d^4y \ e^{iy\cdot(p' + q - p)}$$
$$\times \ \bar u(k') u(k) \ \bar u(p') u(p)$$

$$= (-ig)^2 \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m_\phi^2 + i\epsilon} (2\pi)^4 \delta^4(k' - k - q) (2\pi)^4 \delta^{(4)}(p' - p + q)$$
$$\times \ \bar u(k') u(k) \ \bar u(p') u(p)$$

q integration over 1 δ-fn

$$p - p' = q = k' - k$$

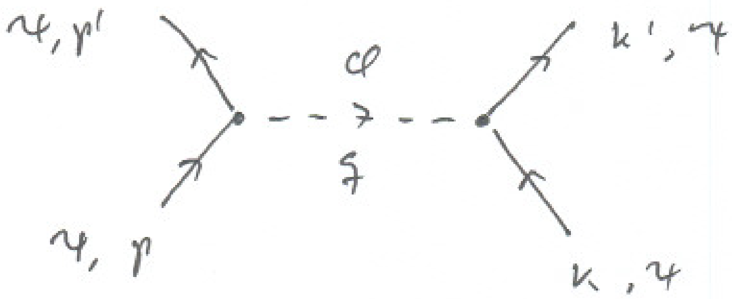
$$= (-ig)^2 \frac{i}{q^2 - m_\varphi^2 + i\epsilon} (2\pi)^4 \delta^{(4)}(\Sigma p)\, \bar{u}(p')u(p)\, \bar{u}(k')u(k)$$

$$= i\mathcal{M}\, (2\pi)^4 \delta(\Sigma p)$$

$$\Rightarrow \quad i\mathcal{M} = \frac{-ig^2}{q^2 - m_\varphi^2}\, \bar{u}(p')u(p)\, \bar{u}(k')u(k) + \ldots$$

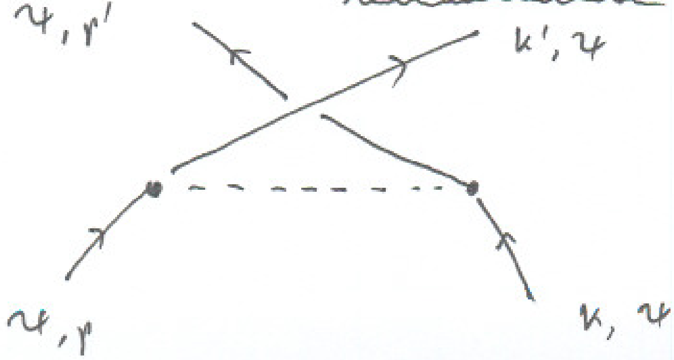
(only 1 type of contraction)

Rather than applying Wick's Theorem, instead we could draw Feynman diagrams:



$\{\mathcal{O}(g^2)\}$

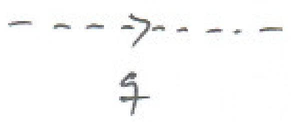
other topologies?

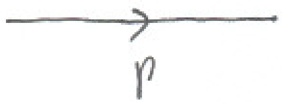


"Crossed Diagram"

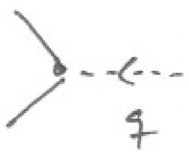
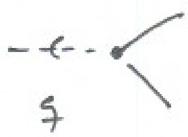
Feynman Rules (Yukawa Theory)

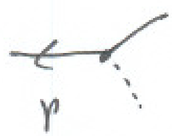
(momentum space rules for $i\mathcal{M}$)

 $= \dfrac{i}{q^2 - m_\varphi^2 + i\epsilon}$

 $= \dfrac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$

 $= -ig$

 $= 1$  $= 1$

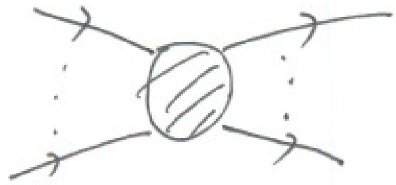
 $= u^s(p)$  $= \bar{u}^s(p)$

(SAME FOR ANTIFERMIONS)

- Impose energy-momentum conservation
- Integrate over loop momentum
- Figure out overall SIGN

{ Note: No symmetry factors! }

Notice that direction of momentum on fermion line is significant.

INITIAL  FINAL

$|p_i\rangle = a^\dagger|0\rangle$ $\qquad$ $\langle p_i| = \langle 0| a$

{ Recall: a_{p_i}, b_{p_i} multiply $e^{-ip\cdot x}$

$a^\dagger_{p_i}, b^\dagger_{p_i}$ " $e^{+ip\cdot x}$ }

Minus signs??

We adopt convention: $|p_i k_i\rangle \hat{=} a^\dagger_{p_i} a^\dagger_{k_i}|0\rangle$

$\langle p_i' k_i'| \hat{=} \langle 0| a_{k_i'} a_{p_i'}$

So $(|p_i k_i\rangle)^\dagger = \langle p_i k_i|$

E.g. Consider some contractions:

(A) $\langle p',k'| (\bar\psi\psi)_x (\bar\psi\psi)_y |p,k\rangle$

$\sim \langle 0| a_{k'} a_{p'} (\bar\psi\psi)_x (\bar\psi\psi)_y a_p^\dagger a_k^\dagger |0\rangle$

$\sim (-1)^2 \langle 0| a_{k'} a_{p'} \bar\psi_y \bar\psi_x \quad \psi_x \psi_y a_p^\dagger a_k^\dagger |0\rangle$

Done!!

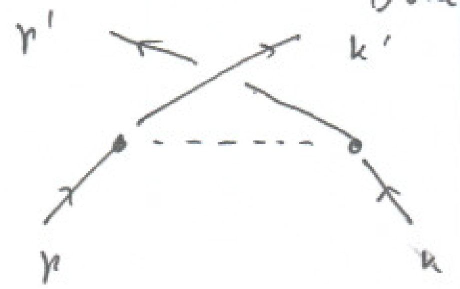
There's another distinct contraction:

(B) $$\langle p',k'|\,(\bar\psi\psi)_x\,(\bar\psi\psi)_y\,|p,k\rangle$$

$$\sim \langle 0|\, a_{k'}\, a_{p'}\, \bar\psi_x\, \psi_x\, \bar\psi_y\, \psi_y\, a_p^\dagger\, a_k^\dagger\, |0\rangle$$

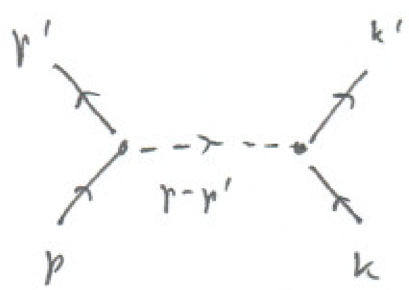
$$\sim (-1)\,\langle 0|\, a_{k'}\, a_{p'}\, \bar\psi_x\, \bar\psi_y\, \psi_x\, \psi_y\, a_p^\dagger\, a_k^\dagger\, |0\rangle$$

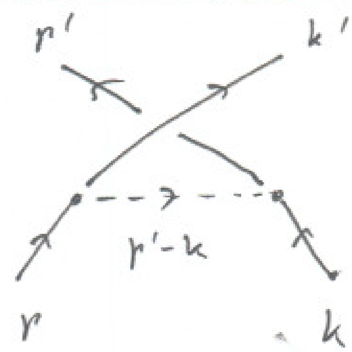
Done !!

$=$ 

CROSSED DIAGRAM

Full Result:

$\boxed{iM =}$ 

$+$ 

$+ \ldots$

(Note: keep external momenta fixed topologically!!)

$$= (-ig)^2 \left\{ \bar u(p')\,u(p)\,\frac{1}{(p-p')^2 - m_\varphi^2}\,\bar u(k')\,u(k) \right.$$

$$\left. - \bar u(p')\,u(k)\,\frac{1}{(p'-k)^2 - m_\varphi^2}\,\bar u(k')\,u(p) \right\}$$

Reflection of Fermi statistics

Note: Closed fermion loop always contains (-1)

i.e. all fields contracted with each other

$$\bar\psi\,\psi\,\bar\psi\,\psi\,\bar\psi\,\psi\,\bar\psi\,\psi$$

$$= (-1)\,\mathrm{tr}\,(S_F\,S_F\,S_F\,S_F)$$

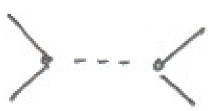
We'll return to issue of converting $\mathcal{M}$ to σ later in context of Q.E.D.

The Yukawa Potential

Let's consider non-relativistic limit of $\psi\psi \to \psi\psi$.

This should give us $\boxed{V(r) = \text{Q.M. potential between 2 heavy } \psi\text{'s}}$

In Q.M. one plugs $V(r)$ into Schrödinger equation to get phase shifts.

If the interacting fermions are distinguishable, only 1st diagram:  contributes.

Recall non-relativistic expansion of energy:

$$E = \sqrt{m^2 + p^2} = m\left(1 + \frac{p^2}{m^2}\right)^{\frac{1}{2}} = m\left(1 + \frac{p^2}{2m^2} + \mathcal{O}(p^4)\right)$$

Hence, in Non-relativistic Limit,

$\underline{p = (E_{p_i}, p_i) \simeq (m, p_i)}$

$\underline{k \simeq (m, k_i)}$ $\qquad$ $\underline{p' \simeq (m, p_i')}$ $\qquad$ $\underline{k' \simeq (m, k_i')}$

(we've neglected terms of $\mathcal{O}(\bar{p}^2, \bar{p}'^2)$)

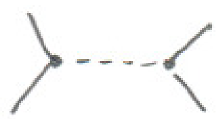
Hence, $\boxed{(p'-p)^2 = -|p_i' - p_i|^2 + \mathcal{O}(p_i^4)}$

$$u^s(p) = \sqrt{m}\begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix} \qquad \left(\xi^{s'\dagger}\xi^s = \delta^{ss'}\right)$$

$$\therefore \quad \boxed{\bar{u}^{s'}(p')\, u^s(p) = 2m\,\delta^{ss'} = \bar{u}^{s'}(k')\, u^s(k)}$$

As expected, spin of each particle is $\underline{\text{separately}}$ conserved.

Put it all together:

$i\mathcal{M} =$  $= -ig^2\, \bar{u}(p')u(p) \dfrac{1}{(p'-p)^2 - m_\phi^2} \bar{u}(k')u(k)$

$$\boxed{i\mathcal{M} \simeq ig^2\, 2m\,\delta^{ss'} \frac{1}{|p_i' - p_i|^2 + m_\phi^2} 2m\,\delta^{rr'} + \ldots}$$

Non-relativistic Limit

Now we can match this result to Born approximation in N.R. Q.M.

$$\boxed{\langle p' | \, iT \, | p \rangle = -i \, \hat{V}(q_i) \, (2\pi) \, \delta(E_{p_i'} - E_{p_i})}$$

$$q_i = p_i' - p_i$$

$$V(x_i) = \int \frac{d^3 q}{(2\pi)^3} \, \hat{V}(q_i) \, e^{i q_i x_i} \qquad (*)$$

Matching $\Rightarrow$ $\boxed{\hat{V}(q_i) = \dfrac{-g^2}{|q_i|^2 + m_\varphi^2}}$

- factors of $2m$ are from relativistic normalization (must be dropped when comparing to N.R. Born App.
- $\delta^{(3)}(p_i - p_i')$ goes away when we integrate over momentum of the target.

Now do Fourier transform $(*)$ to get coordinate space potential:

$$V(x_i) = -g^2 \int \frac{d^3 q}{(2\pi)^3} \, \frac{e^{i q_i x_i}}{|q_i|^2 + m_\varphi^2}$$