Notice that in this form σ depends on the shape of the wave packets.

If initial wave packets are localized in momentum space around p_A, p_B

$$\boxed{\begin{aligned} d\sigma = &\left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \frac{|\mathcal{M}(p_A p_B \to \{p_f\})|^2}{2E_A 2E_B |v_A - v_B|} \\ &\times \int \frac{d^3 k_A}{(2\pi)^3} \int \frac{d^3 k_B}{(2\pi)^3} |\varphi_A(k_{iA})|^2 |\varphi_B(k_{iB})|^2 \\ &\times (2\pi)^4 \delta^{(4)}(k_A + k_B - \Sigma p_f) \end{aligned}}$$

Note:

- $|v_A - v_B| = \left| \frac{\bar{k}_A^z}{\bar{E}_A} - \frac{\bar{k}_B^z}{\bar{E}_B} \right| \quad \left\{ \begin{array}{l} \bar{k}_A^z + \bar{k}_B^z = \Sigma p_f^z \\ \bar{E}_A + \bar{E}_B = \Sigma E_f \end{array} \right\}$

  (Relative velocity of beams as seen from lab frame)

- Usually measurement cannot resolve small variation in momenta:

$$\delta^{(4)}(k_A + k_B - \Sigma p_f) \sim \delta^{(4)}(p_A + p_B - \Sigma p_f)$$

$$\left\{ \text{Then can use } \int \frac{d^3 k}{(2\pi)^3} |\varphi(k_i)|^2 = 1 \right\}$$

localization in momentum gets rid of dependence on shape of wavepacket.

Finally,

$$\boxed{d\sigma = \frac{1}{2E_A 2E_B |v_A - v_B|} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \times |\mathcal{M}(p_A p_B \to \{p_f\})|^2 (2\pi)^4 \delta^{(4)}(p_A + p_B - \Sigma p_f)}$$

CRITICAL RESULT

Note remaining integrals:

$$\boxed{\int d\Pi_n \equiv \left(\prod_f \int \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) (2\pi)^4 \delta^{(4)}(P - \Sigma p_f)}$$

$P \equiv$ total initial 4-momentum

Relativistic n-body phase space

$\mathcal{M}$ is also invariant. Non-invariant piece?

$$\frac{1}{E_A E_B |v_A - v_B|} = \frac{1}{|E_B p_A^z - E_A p_B^z|} = \frac{1}{|\epsilon_{\mu xy\nu} p_A^\mu p_B^\nu|}$$

BOOST INVARIANT ALONG Z-AXIS !!

Therefore $d\sigma$ has Lorentz transformation property of a cross-sectional area !!

Phase space : Special case

2-body final state

p_1, p_2

$$\int d\pi_2 = \int \frac{d^3p_1}{(2\pi)^3}\frac{1}{2E_1}\int\frac{d^3p_2}{(2\pi)^3}\frac{1}{2E_2}(2\pi)^4\,\delta^{(4)}(P-(p_1+p_2))$$

In center-of-mass system $\vec{P}=0$

p_2 integration sets $\vec{p}_2 = -\vec{p}_1$

$E_1 = \sqrt{p_1^2+m_1^2} \qquad E_2 = \sqrt{p_2^2+m_2^2} = \sqrt{p_1^2+m_2^2}$

$$= \int \frac{dp_1\, p_1^2\, d\Omega}{(2\pi)^3\, 2E_1\, 2E_2}(2\pi)\,\delta(E_{cm}-E_1-E_2)$$

total initial energy P_0

Let's do this integral:

$$\left\{ \underline{\text{Recall}}: \quad \delta(f(x)) = \sum_i \frac{1}{\left|\frac{df(x_i)}{dx}\right|} \delta(x - x_i) \quad \text{w/ } f(x_i) = 0 \right\}$$

$$\delta\left(\sqrt{p_1^2 + m_1^2} - E_{cm} + \sqrt{p_1^2 + m_2^2} \right)$$

$$\left\{ \begin{array}{l} f(p_1) \equiv \sqrt{p_1^2 + m_1^2} + \sqrt{p_1^2 + m_2^2} - E_{cm} \\ f'(p_1) = \frac{p_1}{E_1} + \frac{p_1}{E_2} \qquad f(\bar{p}_1) = 0 \end{array} \right\}$$

$$= \left(\frac{p_1}{E_1} + \frac{p_1}{E_2} \right)^{-1} \delta(p_1 - \bar{p}_1)$$

$$\therefore \int d\Pi_2 = \int d\Omega \int dp_1 \frac{p_1^2}{16\pi^2 E_1 E_2} \left(\frac{p_1}{E_1} + \frac{p_1}{E_2} \right)^{-1} \delta(p_1 - \bar{p}_1)$$

$$= \int d\Omega \frac{1}{16\pi^2} \frac{\bar{p}_1}{(E_2 + E_1)}$$

$$\Rightarrow \boxed{\int d\Pi_2 = \int d\Omega \frac{1}{16\pi^2} \frac{|\vec{p}_1|}{E_{cm}}}$$

2-body phase space

If reaction is symmetric about collision axis,

$$\int d\Pi_2 = 2\pi \int d\cos\theta \, \frac{1}{16\pi^2} \frac{|\vec{p}_1|}{E_{cm}}$$

$\Downarrow$

$$\left(\frac{d\sigma}{d\Omega}\right)_{c.m.} = \frac{1}{2E_A 2E_B |v_A - v_B|} \frac{|\vec{p}_1|}{(2\pi)^2 4 E_{c.m.}} \left|\mathcal{M}(p_A p_B \to p_1 p_2)\right|^2$$

What about equal mass case, $m_1 = m_2$??

$$\sqrt{k^2+m^2} + \sqrt{k^2+m^2} = \sqrt{p_1^2+m^2} + \sqrt{p_1^2+m^2}$$

$$E_A + E_B = E_1 + E_2$$

$$k = p_1$$

$$E_A = E_B = E_1 = E_2 = \frac{E_{c.m.}}{2}$$

$$E_A E_B |v_A - v_B| = |E_B p_A^z - E_A p_B^z|$$

$$= \frac{E_{cm}}{2} \cdot 2p_1 = E_{c.m.} p_1$$

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{c.m.} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{c.m.}^2}}$$

Equal mass case

What about differential decay rate $d\Gamma$??

Here we'll state the result:

$$\boxed{d\Gamma = \frac{1}{2M_A}\left(\prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)\left|\mathcal{M}(M_A \to \{p_f\})\right|^2 \times (2\pi)^4\,\delta^4\!\left(p_A - \sum p_f\right)} \quad (\circledast)$$

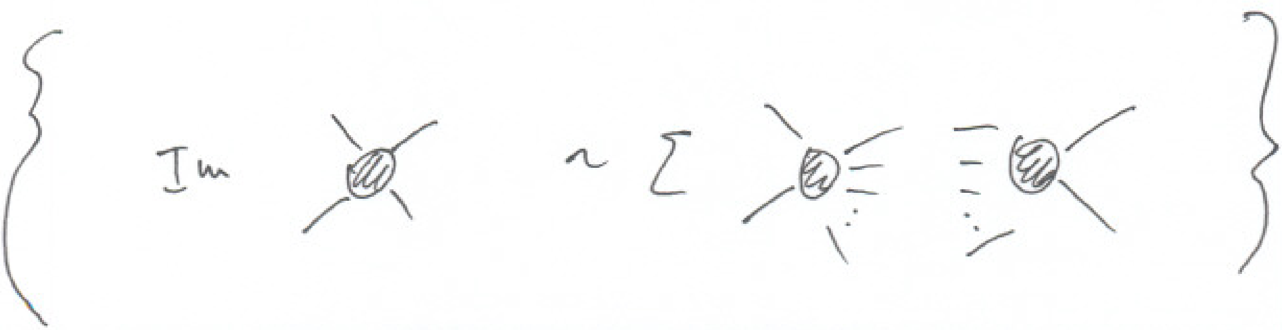
Easy to guess from $d\sigma$:

(A) Remove B factors

(B) A at rest so $2E_A \to 2M_A$

This is subtle physically since unstable particle <u>cannot</u> be sent into distant past.

Derivation of $\circledast$ follows from <u>optical theorem</u>.



S-Matrix from Feynman Diagrams

Want method to compute $\mathcal{M}$ for interacting quantum field theories.

Recall:

$$\langle p_{i1} p_{i2} \cdots | S | k_{iA} k_{iB} \rangle = \lim_{T\to\infty} \langle p_{i1} p_{i2} \cdots | e^{-iH(2T)} | k_{iA} k_{iB} \rangle$$

As in case of 2-pt function, we would like to replace interacting momentum states (eigenstates of H) with eigenstats of H_0.

I.e.:

$$| k_{iA} k_{iB} \rangle \propto \lim_{T\to\infty(1-i\epsilon)} e^{-iHT} | k_{iA} k_{iB} \rangle_0$$

(As we did with vacuum state)

Previously we could argue that vacuum is state with lowest energy. Here have no such argument!!

Very hard to justify!! (proof in 7.2 $\boxed{\text{LSZ Reduction}}$)

We will assume:

(*)
$$\boxed{\langle p_{i1} \dots p_{in} | iT | p_{iA} p_{iB} \rangle = \lim_{T \to \infty(1-i\epsilon)} {}_0\langle p_{i1} \dots p_{in} | T\left\{ \exp\left(-i\int_{-T}^{T} dt\, H_I(t)\right)\right\} | p_{iA} p_{iB} \rangle_0 \Big|_{\substack{\text{CONNECTED} \\ \text{AMPUTATED}}}}$$

will define as we proceed

We want to represent the right-hand side as sum of Feynman diagrams.

Consider perturbative expansion in $\lambda\varphi^4$ theory, with $n = 2$

$$\left\{ |p_i k_i\rangle = \sqrt{2E_{p_i} 2E_{k_i}}\, a^\dagger_{p_i} a^\dagger_{k_i} |0\rangle \right\}$$

At $O(\lambda^0)$ (*) $\Rightarrow$

$${}_0\langle p_{i1} p_{i2} | p_{iA} p_{iB} \rangle_0 = \sqrt{2E_1 2E_2 2E_A 2E_B}\, \langle 0 | a_1 a_2 a_A^\dagger a_B^\dagger | 0 \rangle$$

$$\left\{ \begin{aligned} \underline{\text{Note}}: \quad a_2 a_A^\dagger &= [a_2, a_A^\dagger] + a_A^\dagger a_2 \\ &= (2\pi)^3 \delta^{(3)}(p_{i2} - p_{iA}) + a_A^\dagger a_2 \end{aligned} \right\}$$

$$\therefore \quad \langle 0|a_1 a_2 a_A^\dagger a_B^\dagger|0\rangle = (2\pi)^3 \delta^{(3)}(p_{i2}-p_{iA}) \langle 0|a_1 a_B^\dagger|0\rangle$$

$$+ \langle 0|a_1 a_A^\dagger a_2 a_B^\dagger|0\rangle$$

etc.

$$\boxed{\begin{aligned} {}_0\langle p_{i1}\, p_{i2}|p_{iA}\, p_{iB}\rangle_0 &= 2E_A\, 2E_B\, (2\pi)^6 \\ &\times \Big[\delta^{(3)}(p_{i2}-p_{iA})\, \delta^{(3)}(p_{i1}-p_{iB}) \\ &\quad + \delta^{(3)}(p_{i1}-p_{iA})\, \delta^{(3)}(p_{i2}-p_{iB}) \Big] \end{aligned}} \qquad (*)$$

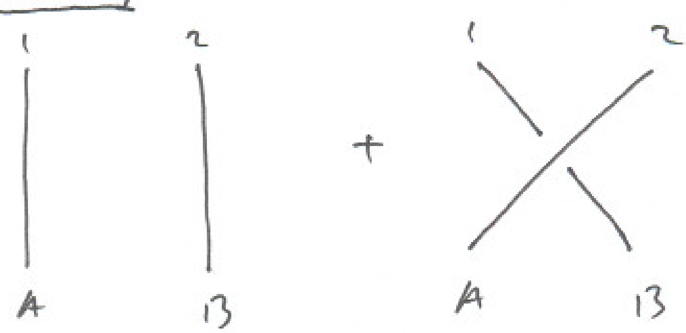
The $\delta^{(3)}$-fns set initial and final momenta to be the same:

$\mathbb{1}$ in $\underline{S = \mathbb{1} + iT}$

as expected (i.e. no interaction)

Hence $(*)$ does not contribute to $\mathcal{M}$.

Diagrammatically:



{ Note: difference with previous analysis is that now have momentum eigenstates. }

At $\underline{\mathcal{O}(\lambda)}$ we have:

$$\boxed{\begin{aligned} &{}_0\langle p_{i1}\, p_{i2} | T\left\{ -\frac{i\lambda}{4!} \int d^4x\, \varphi_I^4(x) \right\} | p_{iA}\, p_{iB} \rangle_0 \\ &\text{(by Wick's Thm)} \\ &= {}_0\langle p_{i1}\, p_{i2} | N\left(-\frac{i\lambda}{4!} \int d^4x\, \varphi_I^4(x) + \text{CONTRACTIONS} \right) | p_{iA}\, p_{iB} \rangle_0 \end{aligned}}$$

Again, difference with previous analysis is that external states are not the vacuum!

$\therefore$ terms that are not fully contracted <u>do not</u> necessarily vanish!!

$$\left\{ \begin{aligned} &\text{Recall free-field decomposition of } \varphi_I(x) = \varphi_I^+(x) + \varphi_I^-(x) \\ &\varphi_I^+(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}}\, a_p\, e^{-ip\cdot x}, \quad \underline{\underline{\varphi_I^+(x)|0\rangle = 0}} \\ &\varphi_I^-(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}}\, a_p^\dagger\, e^{ip\cdot x}, \quad \underline{\underline{\langle 0| \varphi_I^-(x) = 0}} \end{aligned} \right\}$$

Now,

$$\varphi_I^+(x)\, |p_i\rangle_0 = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}}\, a_k\, e^{-ik\cdot x} \sqrt{2E_p}\, a_p^\dagger |0\rangle$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}}\, e^{-ik\cdot x} \sqrt{2E_p}\, (2\pi)^3\, \delta^{(3)}(k_i - p_i)\, |0\rangle$$

$$\Rightarrow \qquad \phi_I^+(x)\,|p_i\rangle_0 = e^{ip\cdot x}\,|0\rangle$$

General structure:

Commute ϕ^+ past a^+ in initial state
commute ϕ^- past a in final state

$\Downarrow$

1 contribution to S-matrix element

Define:

$$\overline{\phi_I(x)\,|p_i\rangle} = e^{-ip\cdot x}$$
$$\overline{\langle p_i|\,\phi_I(x)} = e^{ip\cdot x}$$

To get S-matrix:

write down all possible contractions of $\phi_I(x)$ operators with external momenta.

Is this correct?

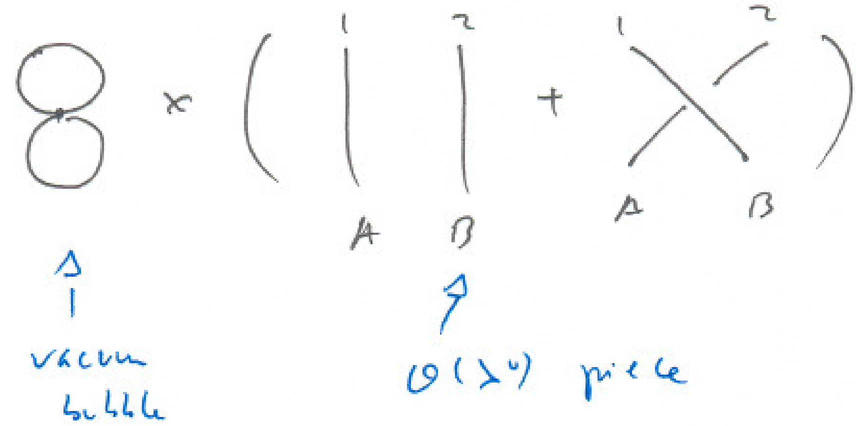
At $O(\lambda)$ have terms of form:

$$\phi\phi\phi\phi \qquad \overline{\phi\phi}\,\phi\phi \qquad \overline{\phi\phi}\;\overline{\phi\phi}$$

Consider fully contracted contribution:

$$-\frac{i\lambda}{4!}\int d^4x \; {}_0\langle p_{f1}\, p_{f2} |\, \overbrace{\phi\phi}\;\overbrace{\phi\phi}\, | p_{iA}\, p_{iB}\rangle_0$$

$=$



Again this contributes to $\mathbb{1}$ in S!!

$$-\frac{i\lambda}{4!}\int d^4x \; {}_0\langle p_{f1}\, p_{f2} |\, \overbrace{\phi\phi}\;\phi\phi\, | p_{iA}\, p_{iB}\rangle$$

Normal ordered $\Rightarrow$ $a^\dagger a^\dagger + 2a^\dagger a + aa$

{ In moving these operators past a's and $a^\dagger$'s of initial and final state momenta, only terms w/ equal number of a's and $a^\dagger$'s survive. }

$\Downarrow$

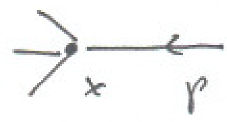
one	ϕ	gets	contracted	w/	initial state	$\lvert p_i\rangle$	
"	"	"	"	"	final	"	"

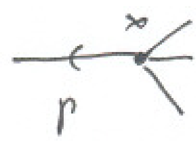
This leaves uncontracted $|p_i\rangle$ and $\langle p_f|$

$\Downarrow$

δ-fn

Contractions diagrammatically:

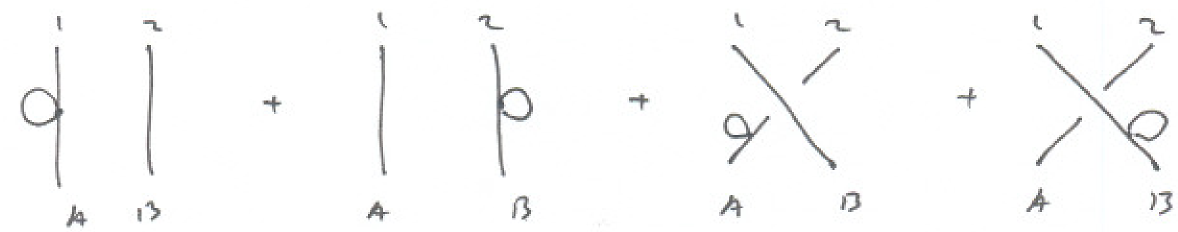
$\phi_I(x) | p_i \rangle =$ 

$\langle p_i | \phi_I(x) =$ 

Of course, Feynman diagrams for S-matrix elements always contain external lines.

Putting it all together:

$$-\frac{i\lambda}{4!}\int d^4x \; {}_0\langle p_1 p_2 | \phi\phi\phi\phi | p_A p_B \rangle$$

$=$ 

Again trivial !! ($\mathbb{1}$ in S)

Only fully-connected diagrams in which all external lines are connected to each other contribute to T.

Consider the final contribution:

$$-\frac{i\lambda}{4!}\int d^4x \; {}_0\langle p_1\, p_2 |\, \varphi\,\varphi\,\varphi\,\varphi \,| p_A\, p_B \rangle$$

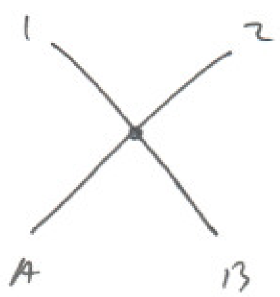
totally uncontracted (w/ each other)

$$\boxed{\begin{array}{l}\text{Contract } 2\ \varphi\text{'s with } |p_A\, p_B\rangle_0 \ + \\ \quad\quad 2\ \varphi\text{'s with } {}_0\langle p_1\, p_2|\end{array}}$$

4! ways to do this

{ e.g. $\langle p_1 p_2 |\, \varphi\ \varphi\ \varphi\ \varphi \,| p_A\, p_B\rangle$ etc. }

One finds:



$$= (4!)\left(-\frac{i\lambda}{4!}\right)\int d^4x \; e^{-i(p_A+p_B-p_1-p_2)\cdot x}$$

$$= -i\lambda\,(2\pi)^4\,\delta^{(4)}(p_A+p_B-p_1-p_2)$$

$$= i\mathcal{M}\,(2\pi)^4\,\delta^{(4)}(p_A+p_B-p_1-p_2)$$

$$\boxed{\mathcal{M} = -\lambda}$$

Aside: $|\mathcal{M}|^2 = \lambda^2$

$$\left(\frac{d\sigma}{d\Omega}\right)_{c.m.} = \frac{\lambda^2}{64\pi^2 E_{c.m.}^2}$$

$$\Rightarrow \boxed{\sigma = \frac{\lambda^2}{32\pi E_{c.m.}^2}}$$

can use to extract λ and then predict some other process

What other sorts of diagrams do we expect?

$\langle p_1 p_2 | iT | p_A p_B \rangle \sim$ 

disconnecteds $\Rightarrow$ normalized away

$\hookrightarrow$ more problematic..

Let's investigate..