The S-Matrix

How do we describe interactions of particles and fields (with an eye to measurement)??

(A) Set up wave packets representing initial-state particles.

(B) Evolve corresponding state with e^{-iHt} where H is full, interacting Hamiltonian.

(C) Overlap resulting final state w/ wave packets representing set of final-state particles.

Overlap $\Rightarrow$ Probability amplitude for producing that state

Wave packet:

$$|\varphi\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{k_i}}} \varphi(k_i) |k_i\rangle$$

$\left\{ \begin{array}{l} \varphi(k_i) = \text{Fourier transform of spatial wave function} \\ |k_i\rangle = \text{one-particle state of momentum } k_i \text{ in } \underline{\text{interacting}} \text{ theory.} \end{array} \right\}$

$$\langle \varphi | \varphi \rangle = \int \frac{d^3k\, d^3p}{(2\pi)^6} \frac{1}{\sqrt{2E_{k_i} 2E_{p_i}}} \varphi(k_i)\varphi(p_i) \underbrace{\langle p_i | k_i \rangle}_{2E_{k_i}(2\pi)^3 \delta^3(p_i - k_i)}$$

$$= \underbrace{1 = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_{k_i}} |\varphi(k_i)|^2}$$

Probability that we want is:

$$\boxed{P = |\langle \varphi_1 \varphi_2 \cdots | \varphi_A \varphi_B \rangle|^2}$$

FINAL STATE (OUT-STATE)

INITIAL STATE (IN-STATE)

$\left\{ \right.$ NOTE:

- Only will consider $2 \longrightarrow$ anything scattering.
- Heisenberg picture so states are time-independent $\left. \right\}$

Let's take $|\varphi_A \varphi_B\rangle_{IN}$ as superposition

of states of sharp momenta :

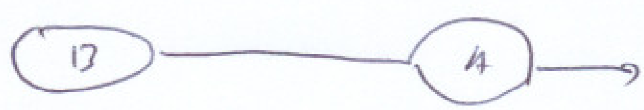
$$|p_{iA}\ p_{iB}\rangle$$

$$|\varphi_A \varphi_B\rangle_{IN} = \int \frac{d^3k_A}{(2\pi)^3} \frac{d^3k_B}{(2\pi)^3} \frac{\varphi_A(k_{iA})\varphi_B(k_{iB})}{\sqrt{2E_A\, 2E_B}}\, e^{-i b_i k_{iB}}\, |k_{iA}\, k_{iB}\rangle$$

IMPACT PARAMETER b

wave packets φ_A and φ_B can be displaced relative to one another :

We choose convention such that φ_A and φ_B are collinear



and put in $e^{-i b_i k_{iB}}$ to account for spatial translation.

Similarly we can define the "OUT" state.

$$\underbrace{{}_{\text{OUT}}\langle \phi_1 \phi_2 \ldots | = \left(\prod_f \int \frac{d^3 p_f}{(2\pi)^3} \frac{\phi_f(p_{if})}{\sqrt{2E_f}} \right) {}_{\text{OUT}}\langle p_{i1} p_{i2} \ldots |}$$

{ Applies to everything in parenthesis }

{ It is easier to use "OUT" states of sharp momenta (rather than superpositions). Reasonable as long as measuring apparatus does not resolve positions! }

We can now relate probability of scattering in a real experiment to our idealized transition amplitude:

$$\boxed{\begin{aligned} &{}_{\text{OUT}}\langle p_{i1} p_{i2} \ldots | k_{iA} k_{iB} \rangle_{\text{IN}} \\ &= \lim_{T\to\infty} \langle p_{i1} p_{i2} \ldots ; T | k_{iA} k_{iB} ; -T \rangle \\ &= \lim_{T\to\infty} \langle p_{i1} p_{i2} \ldots | e^{-iH(2T)} | k_{iA} k_{iB} \rangle \end{aligned}}$$

Defined at common reference time

Hence IN and OUT states are related by limit of a sequence of unitary operators.

Therefore, define the S-MATRIX

$$\boxed{{}_{\text{OUT}}\langle p_1 p_2 \ldots | k_A k_B \rangle_{\text{IN}} \equiv \langle p_1 p_2 \ldots | S | k_A k_B \rangle}$$

If there is no interaction, $S = \mathbb{1}$

$\therefore$ can isolate the non-trivial piece

$$\boxed{S = \mathbb{1} + iT}$$

T-MATRIX

We should always have energy-momentum conservation. Therefore, define:

$$\boxed{\langle p_1 p_2 \ldots | iT | k_A k_B \rangle = (2\pi)^4 \delta^4 \left(k_A + k_B - \sum p_f\right) \times i\mathcal{M}(k_A k_B \rightarrow p_f)}$$

Note:

- All particles are "on-the-mass-shell"
  $p^0 = E_{p}$ $k^0 = E_k$
- This definition "kinematics" from "dynamics"
- $\mathcal{M}$ is what we compute using QFT!!

The Cross Section

Recall from N.R. Q.M.

Scattering wave function at large distances from scattering center:

$$\psi^{(+)}(r_i) \propto e^{ik_i r_i} + \frac{e^{ikr}}{r} f(\theta, \varphi)$$

$$\frac{d\sigma}{d\Omega} = |f(\theta,\varphi)|^2 \qquad \frac{\#\text{ of particles detected / second}}{\text{initial flux of particles}}$$

{ flux = # of particles passing through unit area transverse to beam per second }

Cross section gives likelihood of any particular final state

SET UP

Consider collision of two beams of particles with well-defined momenta and see what comes out.

Cross section is intrinsic to colliding particles. Allows comparison between different experiments.

length of bunch of particles

particles/volume

$$\boxed{\sigma = \frac{\text{Number of scattering events of particular type}}{\rho_A \, l_A \, \rho_B \, l_B \, A}}$$

cross-sectional area common to both bunches

Note:

- Symmetric between A and B
- $\sigma \sim$ area $\left(\sim \dfrac{\#}{\frac{A^2}{V^2}} \sim \dfrac{A^2 l^2}{A^2} \sim l^2 \right)$
- ρ_A, ρ_B not generally constant

$\Downarrow$

Number of events $= \sigma \, l_A \, l_B \int d^2x \, \rho_A(x) \rho_B(x)$

Assume constant densities $\Rightarrow$

Number of events $= \sigma \, l_A l_B \, \rho_A \rho_B \, A$

$$" \quad = \sigma \frac{N_A N_B}{A}$$

total #'s of particles of type A, B

Many different final states can be relevant to single scattering experiment:

E.g. $e^+e^- \to e^+e^-$ 2-body elastic

$\to \mu^+\mu^-$ "

$\to \tau^+\tau^-$ "

$\to \mu^+\mu^-\gamma$ 3-body

$\to \mu^+\mu^-\gamma\gamma$ 4-body

Useful to define a differential cross section:

(*) $$\boxed{\frac{d\sigma}{d(d^3p_1 \cdots d^3p_n)}}$$

region of final state momentum space

{Note: final state momenta are constrained by 4-momentum conservation}

INTEGRATE (*) to get cross section

For 2-body elastic scattering, $\underline{n=2}$

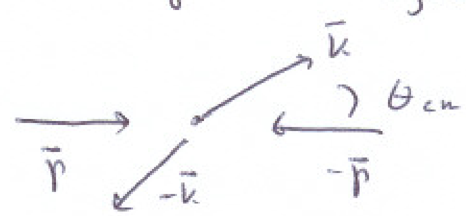
2 unconstrained momentum components $\Rightarrow$

angles $\underline{\underline{\theta \text{ and } \phi}}$ of momentum of one of particles

$\therefore$

$$\frac{d\sigma}{d(d^3p_1 \ldots d^3p_n)} \xrightarrow[\text{conservation}]{\text{momentum}} \boxed{\frac{d\sigma}{d\Omega}}$$

E.g. in center-of-mass system



Decay Rate

$\Gamma \equiv$ decay rate of unstable particle A AT REST

$$= \frac{\text{Number of decays per unit time}}{\text{Number of A particles present}}$$

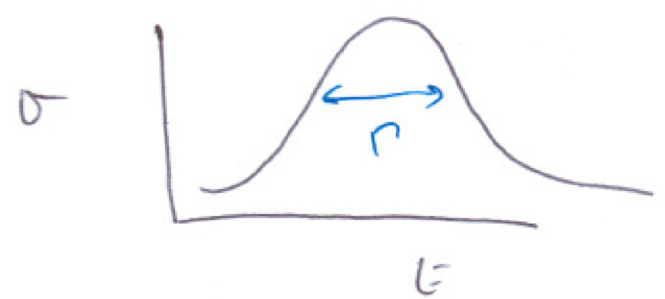
$$\tau = \text{lifetime} = \Big(\sum_{\substack{\text{ALL}\\ \text{FINAL}\\ \text{STATES}}} \Gamma \Big)^{-1}$$

In N.R. Q.M. a resonance shows up as a pole in the scattering amplitude:

$$f(E) \propto \frac{1}{E - E_0 + \frac{i\Gamma}{2}}$$

$$\therefore \quad \sigma \sim |f|^2 \propto \frac{1}{(E-E_0)^2 + \frac{\Gamma^2}{4}}$$

So width of resonant peak in σ is decay rate of unstable state.



In Q.F.T. might expect:

$$\frac{1}{p^2 - m^2 + im\Gamma} \qquad ??$$

How does propagator get modified??

Let's consider some simple diagrams.

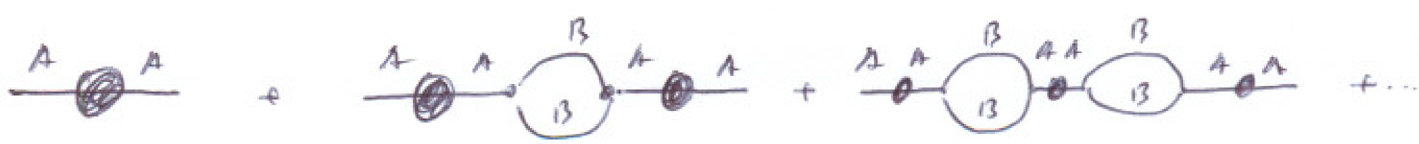
We can think of the mass in the propagator as arising from self-interactions of a massless particle:

$$= \frac{1}{p^2-m^2} = \frac{1}{p^2} + \frac{1}{p^2}\, m^2 \,\frac{1}{p^2} + \frac{1}{p^2}\, m^2 \,\frac{1}{p^2}\, m^2 \,\frac{1}{p^2} + \dots$$

$$= \frac{1}{p^2}\left(1 + \frac{m^2}{p^2} + \left(\frac{m^2}{p^2}\right)^2 + \dots\right)$$

$$= \frac{1}{p^2\left(1 - \frac{m^2}{p^2}\right)} = \frac{1}{p^2-m^2}$$

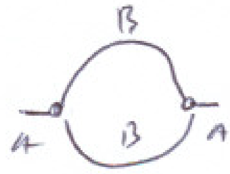
Now consider more complicated self interaction:

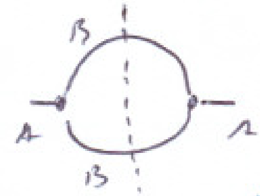


$$\frac{1}{p^2-m^2} + \frac{1}{p^2-m^2}\, f(p^2) \,\frac{1}{p^2-m^2} + \frac{1}{p^2-m^2}\, f(p^2) \,\frac{1}{p^2-m^2}\, f(p^2) \,\frac{1}{p^2-m^2} + \dots$$

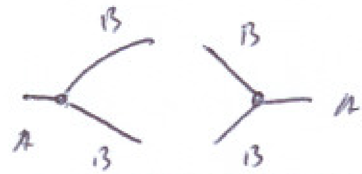
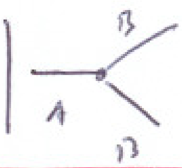
$$= \frac{1}{p^2-m^2-f(p^2)} \xrightarrow[p^2\sim m^2]{} \frac{1}{p^2-\hat{m}^2 - i\,\mathrm{Im} f(m^2)}$$

$$\left\{ \hat{m}^2 = m^2 + \mathrm{Re} f(m^2) \right\}$$

$f(p^2) =$ 

$\text{Im}\, f(m^2) =$ 

Intermediate states "on shell"

$$\therefore \quad \boxed{\text{Im}\, f(m^2) \sim}$$  $\sim \Big|$  $\Big|^2 \sim \Gamma$

decay of A to BB

Let's now return to our general formula for the T-matrix:

$$\langle p_{i1}, p_{i2}, \ldots \,|\, iT \,|\, k_{iA} k_{iB} \rangle = (2\pi)^4 \delta^4(k_A + k_B - \Sigma p_f)$$
$$\times \; i\mathcal{M}(k_A k_B \to p_f)$$

How do we construct σ from $\mathcal{M}$??

{ Recall that $\mathcal{M}$ is the "dynamical" part of S-matrix that we will obtain using Feynman diagrams }

Consider probability for initial state $(\varphi_A \varphi_B)$ to scatter and become final state of n particles whose momenta lie in a small region $d^3p_1 \cdots d^3p_n$:

$$P(AB \to 1,2,\ldots,n) = \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f}\right) \left| {}_{OUT}\langle p_1 \cdots p_n | k_A k_B \rangle_{IN} \right|^2$$

For single target A particle and many incident B particles with different impact parameters b_i,

$$\boxed{\# \text{ of scattering events} = N = \sum_{\substack{\text{ALL INCIDENT}\\ \text{PARTICLES } i}} P_i = \int d^2b \, n_B \, P(b_i)}$$

density of B particles per unit AREA

Assume n_B doesn't vary over range of interaction

$$\boxed{N = n_B \int d^2 b \; P(b_i)}$$

Recall:

$$\boxed{\sigma = \frac{N}{N_A N_B / A} = \frac{N}{1 \cdot n_B} = \int d^2b \; P(b_i)}$$

Now we can get expression for σ in terms of $\mathcal{M}$!!

We have (infinitesimal) cross-section for scattering in small region of momenta $d^3p_1 \ldots d^3p_n$:

$$d\sigma = \left(\prod_f \frac{d^3p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \int d^2b \, \left| {}_{\text{OUT}}\langle p_1 \ldots p_n | \phi_A \phi_B(b) \rangle_{\text{IN}} \right|^2$$

wave packets

Now recall:

$$|\phi_A \phi_B(b)\rangle_{\text{IN}} = \int \frac{d^3k_A}{(2\pi)^3} \frac{d^3k_B}{(2\pi)^3} \frac{\phi_A(k_{iA})\phi_B(k_{iB})}{\sqrt{(2E_A)(2E_B)}} e^{-ib_i k_{iB}} |k_{iA} k_{iB}\rangle_{\text{IN}}$$

Plugging this in gives:

$$\boxed{\begin{aligned} d\sigma = & \left(\prod_f \frac{d^3p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \int d^2b \left(\prod_{i=A,B} \int \frac{d^3k_i}{(2\pi)^3} \frac{\phi_i(k_i)}{\sqrt{2E_i}} \int \frac{d^3\bar{k}_i}{(2\pi)^3} \frac{\phi_i^*(\bar{k}_i)}{\sqrt{2\bar{E}_i}} \right) \\ & \times e^{ib_i(\bar{k}_{iB} - k_{iB})} \\ & \times \left({}_{\text{OUT}}\langle \{p_f\} | \{k_i\} \rangle_{\text{IN}} \right) \left({}_{\text{OUT}}\langle \{p_f\} | \{\bar{k}_i\} \rangle_{\text{IN}} \right)^* \end{aligned}}$$

(Note: horrible notation ...)

Note:

- $$\int d^2b \, e^{i\vec{b}\cdot(\vec{\bar{k}}_B^\perp - \vec{k}_B^\perp)} = (2\pi)^2 \delta^{(2)}(\vec{\bar{k}}_B^\perp - \vec{k}_B^\perp)$$

- Drop $\mathbb{1}$ in $S = \mathbb{1} + iT$

- $${}_{\text{out}}\langle \{p_f\} | \{k_i\} \rangle_{\text{in}} = (2\pi)^4 \delta^{(4)}(\Sigma k_i - \Sigma p_f) \times i\underline{\mathcal{M}(\{k_i\} \to \{p_f\})}$$

- $$\left({}_{\text{out}}\langle \{p_f\} | \{\bar{k}_i\} \rangle_{\text{in}} \right)^* = -(2\pi)^4 \delta^{(4)}(\Sigma \bar{k}_i - \Sigma p_f) \times i\underline{\mathcal{M}(\{\bar{k}_i\} \to \{p_f\})}$$

Can use these δ-fns to perform all 6 $\bar{k}_A, \bar{k}_B$ integrals !!

(Do as exercise)

Finally,

$$\boxed{\begin{aligned} d\sigma = \left(\prod_f \frac{d^3p_f}{(2\pi)^3} \frac{1}{2E_f} \right) & \frac{|\mathcal{M}(p_A p_B \to \{p_f\})|^2}{2E_A 2E_B |v_A - v_B|} \\ & \times \int \frac{d^3k_A}{(2\pi)^3} \int \frac{d^3k_B}{(2\pi)^3} |\varphi_A(k_{iA})|^2 |\varphi_B(k_{iB})|^2 \\ & \times (2\pi)^4 \delta^{(4)}(k_A + k_B - \Sigma p_f) \end{aligned}}$$