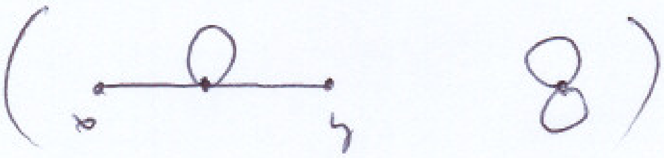
Exponentiation of Disconnected Diagrams

A typical diagram contributing to 2-pt function:
(numerator only):



$\left\{ \sim D_F(x-y)\int d^4z \, D_F^2(z-z) \right\}$

"connected" "disconnected" (from x and y)

Say $V_i \in \{$disconnected pieces$\}$

e.g. z

n_i = # of pieces of form V_i in generic diagram.

If V_i is also "value" of piece V_i, then

$$\boxed{\text{generic diagram} = (\text{value of connected piece}) \prod_i \frac{1}{n_i!}(V_i)^{n_i}}$$

Symmetry factor from interchanging n_i copies of V_i.

$\langle \Omega | T\{\varphi(x)\varphi(y)\}|\Omega\rangle$ (connected only) $=$ SUM OF ALL DIAGRAMS

$$= \sum_{\substack{\text{All possible}\\ \text{CONNECTED}\\ \text{PIECES}}} \sum_{\text{ALL } \{n_i\}} \left(\begin{matrix}\text{VALUE OF}\\ \text{CONNECTED}\\ \text{PIECE}\end{matrix}\right) \prod_i \frac{1}{n_i!}(V_i)^{n_i}$$

$\{n_i\}$ = ordered sets $\{n_1, n_2, n_3, \ldots\}$ of non-negative integers

$$= \left(\sum \text{connected}\right) \times \sum_{\text{ALL } \{n_i\}} \left(\prod_i \frac{1}{n_i!}(V_i)^{n_i}\right)$$

$$= \quad " \quad \times \left(\sum_{n_1} \frac{1}{n_1!} V_1^{n_1}\right)\left(\sum_{n_2} \frac{1}{n_2!} V_2^{n_2}\right)\cdots$$

$$= \quad " \quad \prod_i \left(\sum_{n_i} \frac{1}{n!} V_i^{n_i}\right)$$

$$= \quad " \quad \prod_i \exp(V_i)$$

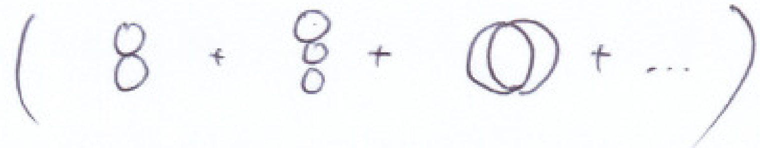
$$= \left(\sum \text{connected}\right) \times \exp\left(\sum_i V_i\right) =$$ SUM OF ALL DIAGRAMS

"Factorization"

In terms of diagrams:

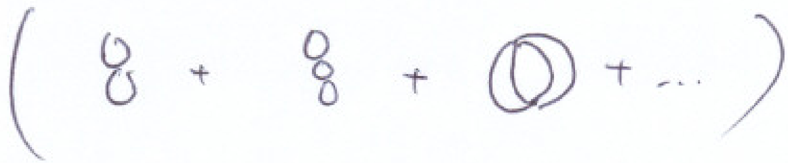
$$\lim_{T \to \infty(1-i\epsilon)} \langle 0 | T\left\{ \varphi_I(x)\, \varphi_I(y) \exp\left[-i \int_{-T}^{T} dt\, H_I(t) \right] \right\} | 0 \rangle$$

$=$ 

$\times \exp$ 

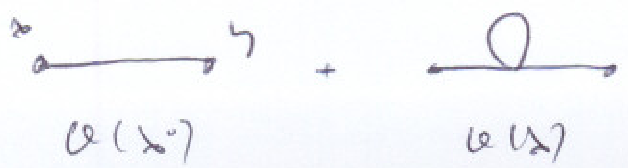
By identical argument,

$$\langle 0 | T\left\{ \exp\left[-i \int_{-T}^{T} dt\, H_I(t) \right] \right\} | 0 \rangle$$

$= \exp$ 

Therefore,

$\boxed{\langle \Omega | T\{ \varphi(x)\, \varphi(y) \} | \Omega \rangle = \text{SUM OF ALL CONNECTED DIAGRAMS WITH EXTERNAL POINTS } x \text{ \& } y.}$

$=$ 

$+ \mathcal{O}(\lambda^2)$

what is physical interpretation of the disconnected diagrams ??

Recall that (before normalization)

$$\langle \Omega | T\{\varphi(x)\varphi(y)\} | \Omega\rangle \lim_{T\to\infty(1-i\epsilon)} \left(|\langle 0|\Omega\rangle|^2 e^{-iE_0(2T)} \right)$$

$$= \lim_{T\to\infty(1-i\epsilon)} \langle 0 | T\{ \varphi_I(x)\varphi_I(y) e^{-i\int_{-T}^{T} dt\, H_I(t)} \} | 0\rangle$$

$$= \left(\sum \text{connected}\right) e^{(\sum \text{disconnected})}$$

Match T dependent pieces ⇒

$$e^{\sum_i V_i} \propto e^{-iE_0(2T)}$$

$$\left\{ \text{Recall } (2\pi)^4 \delta^4(0) = 2T\cdot V \right\}$$

$$\Rightarrow \quad \frac{\sum_i V_i}{(2\pi)^4\delta^4(0)} = -i \frac{E_0(2T)}{2T\cdot V}$$

$\Rightarrow$

$\dfrac{E_0}{\text{VOLUME}} = i$  $\overline{\ (2\pi)^4\,\delta^4(0)\ }$

Independent of T & L

Energy density of the vacuum

$\{$ Relative to zero of energy set by $H_0|0\rangle = 0$ $\}$

Generalization of all of this to other correlators is easy!

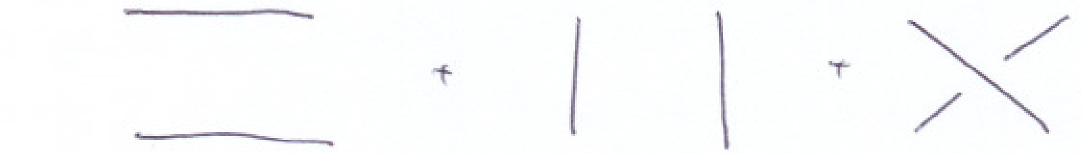
(*) $\boxed{\langle\Omega|T\{\phi(x_1)\ldots\phi(x_n)\}|\Omega\rangle = \text{SUM OF ALL CONNECTED DIAGRAMS WITH } n \text{ EXTERNAL POINTS}}$

(Have same exponentiation and cancellation of disconnected diagrams)

(*) One of most important results in Q.F.T.

Example (of interest to us)

$$\langle \Omega | T\{ \varphi_1 \varphi_2 \varphi_3 \varphi_4 \} | \Omega \rangle =$$



$O(\lambda^0)$

$O(\lambda)$

$O(\lambda^2)$

Note that in some diagrams points are disconnected from each other.

But these don't factor & exponentiate ...