Wick's Theorem

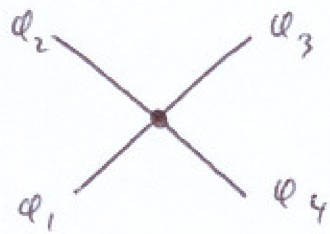
In Q.F.T. relevant objects are correlation functions:

$$\langle 0 | T \{ \varphi_I(x_1) \varphi_I(x_2) \ldots \varphi_I(x_n) \} | 0 \rangle \quad (*)$$

($n=2$ $\Rightarrow$ Feynman propagator)

$n>2$ can evaluate by brute force, plugging in Fourier expansions of fields in terms of ladder operators.

$n=4$ $\Rightarrow$ Scattering



This will be of special relevance to us.

Let's learn some simplifying tricks for evaluating $(*)$.

Strategy: Move destruction operators step-by-step to the right; they will then vanish when acting on vacuum state: "Normal ordering a time-ordered product"

Consider

$$\langle 0 | T\{\varphi_I(x)\,\varphi_I(y)\} | 0 \rangle$$

(We already know how to calculate this)

Say $\boxed{\varphi_I(x) = \varphi_I^+(x) + \varphi_I^-(x)}$ (+ and − frequency parts)

$$\varphi_I^+(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_{p_i} e^{-ip\cdot x}, \qquad \varphi_I^-(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_{p_i}^+ e^{ip\cdot x}$$

$\Big\{$ <u>Recall</u>: $T\,\varphi(x)\varphi(y) = \Theta(x^0 - y^0)\,\phi(x)\varphi(y) + \Theta(y^0 - x^0)\,\varphi(y)\varphi(x)$ $\Big\}$

Note that: $\varphi_I^+(x)|0\rangle = 0 \qquad \langle 0|\varphi_I^-(x) = 0$

Consider $x^0 > y^0$

$$T\{\varphi_I(x)\varphi_I(y)\} = \varphi_I(x)\varphi_I(y)$$

$$= \varphi_I^+(x)\varphi_I^+(y) + \varphi_I^+(x)\varphi_I^-(y)$$
$$+ \varphi_I^-(x)\varphi_I^+(y) + \varphi_I^-(x)\varphi_I^-(y)$$

$$= \varphi_I^+(x)\varphi_I^+(y) + \varphi_I^-(y)\varphi_I^+(x) + [\varphi_I^+(x), \varphi_I^-(y)]$$
$$+ \varphi_I^-(x)\varphi_I^+(y) + \varphi_I^-(x)\varphi_I^-(y)$$

$\Big\{$ Note that +'s (i.e. a_{p_i}'s) are to the right of all −'s (i.e. $a_{p_i}^+$'s) <u>except</u> for commutator. $\Big\}$

Normal Ordered

$a^\dagger a^\dagger a a$, $a^\dagger a$, etc

Define

$$\boxed{N(a_1 a_2^\dagger a_3) \equiv a_2^\dagger a_1 a_3}$$

order irrelevant as they commute

Define contraction of two fields:

$$\boxed{\overbracket{\varphi(x)\,\varphi}(y) = \begin{cases} [\varphi^+(x), \varphi^-(y)] & x^0 > y^0 \\ [\varphi^+(y), \varphi^-(x)] & y^0 > x^0 \end{cases}}$$

(Drop I subscript)

Note that:

$$\boxed{\overbracket{\varphi(x)\,\varphi}(y) = D_{12}(x-y)}$$

Recall:

$x^0 > y^0$

$T\{\varphi(x)\varphi(y)\} =$ normal ordered products $+ [\varphi^+(y), \varphi^-(x)]$

$y^0 > x^0$

$\qquad$ " $=$ " $+ [\varphi^+(x), \varphi^-(y)]$

$$\boxed{T\{\varphi(x)\varphi(y)\} = N\{\varphi(x)\varphi(y) + \overbracket{\varphi(x)\,\varphi}(y)\}}$$

2-point fn

Easily generalized to n-point function:

$$\boxed{T\{\varphi(x_1)\varphi(x_2)\dots\varphi(x_n)\} = N\{\varphi(x_1)\varphi(x_2)\dots\varphi(x_n) + \text{All possible contractions of fields}\}}$$

1 term for each way of contracting

Wick's Theorem

$n=4$ (note compact notation $\varphi_i = \varphi(x_i)$)

$$\begin{aligned} T\{\varphi_1\varphi_2\varphi_3\varphi_4\} = N\{ & \varphi_1\varphi_2\varphi_3\varphi_4 \\ & + \varphi_1^{\bullet}\varphi_2^{\bullet}\varphi_3\varphi_4 + \varphi_1^{\bullet}\varphi_2\varphi_3^{\bullet}\varphi_4 + \varphi_1^{\bullet}\varphi_2\varphi_3\varphi_4^{\bullet} \\ & + \varphi_1\varphi_2^{\bullet}\varphi_3^{\bullet}\varphi_4 + \varphi_1\varphi_2^{\bullet}\varphi_3\varphi_4^{\bullet} + \varphi_1\varphi_2\varphi_3^{\bullet}\varphi_4^{\bullet} \\ & + \varphi_1^{\bullet}\varphi_2^{\bullet}\varphi_3^{\circ}\varphi_4^{\circ} + \varphi_1^{\bullet}\varphi_2^{\circ}\varphi_3^{\bullet}\varphi_4^{\circ} + \varphi_1^{\bullet}\varphi_2^{\circ}\varphi_3^{\circ}\varphi_4^{\bullet}\} \end{aligned}$$

$\{$ E.g. $N\{\varphi_1^{\bullet}\varphi_2\varphi_3^{\bullet}\varphi_4\} \Rightarrow D_F(x_1 - x_3)\cdot N\{\varphi_2\varphi_4\}$ $\}$

Crucial point: $\boxed{\langle 0|\, N\{\text{anything}\}\,|0\rangle = 0}$

Only fully contracted terms survive!!

Therefore,

$$
\begin{aligned}
\langle 0 | T\{\varphi_1 \varphi_2 \varphi_3 \varphi_4\} | 0 \rangle = \; & \varphi_1 \varphi_2 \varphi_3 \varphi_4 \\
& + \varphi_1 \varphi_2 \varphi_3 \varphi_4 \\
& + \varphi_1 \varphi_2 \varphi_3 \varphi_4
\end{aligned}
$$

$$
\Rightarrow \boxed{
\begin{aligned}
\langle 0 | T\{\varphi_1 \varphi_2 \varphi_3 \varphi_4\} | 0 \rangle = \; & D_F(x_1 - x_2)\, D_F(x_3 - x_4) \\
& + D_F(x_1 - x_3)\, D_F(x_2 - x_4) \\
& + D_F(x_1 - x_4)\, D_F(x_2 - x_3)
\end{aligned}
}
$$

{ For proof of Wick's Thm (by induction) see P + S and Bjorken + Drell. }

GRAPHICAL REPRESENTATION:

FEYNMAN DIAGRAMS

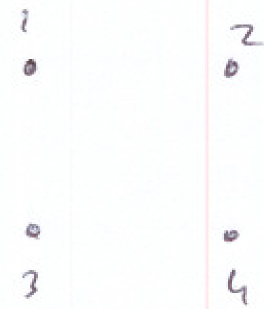
Wick's Theorem turns correlators of form

$$\langle 0 | T\{ \varphi_I(x_1)\, \varphi_I(x_2) \ldots \varphi_I(x_n) \} | 0 \rangle$$

into sums of products of Feynman propagators.

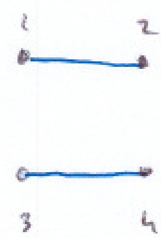
Consider case of 4 fields:

Diagramatically:



• = space-time point

$\langle 0 | T\{\varphi_1 \varphi_2 \varphi_3 \varphi_4\} | 0 \rangle =$



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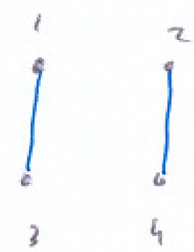

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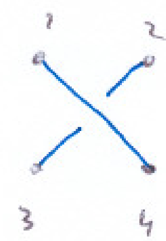


$D(x_1-x_2)\,D(x_3-x_4) + D(x_1-x_3)\,D(x_2-x_4) + D(x_1-x_4)\,D(x_2-x_3)$

"Feynman Diagram":

Particle "creation", "propagation", "destruction"

{ 3 ways this can happen: 3 ways to connect points in pairs }

Let's consider more complicated examples.

Recall: general perturbative expansion for 2-pt function

$$\langle \Omega | T\{\varphi(x)\varphi(y)\} | \Omega \rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0 | T\{\varphi_I(x)\varphi_I(y)\, e^{-i\int_{-T}^{T} dt\, H_I(t)}\} | 0 \rangle}{\langle 0 | T\{ e^{-i\int_{-T}^{T} dt\, H_I(t)}\} | 0 \rangle}$$

Let's focus on this expression.

(ignore denominator: related to normalization)

Expand numerator in perturbation theory:

$$\langle 0|\, T\{\varphi(x)\varphi(y) + \varphi(x)\varphi(y)\left(-i\int dt\, H_I(t)\right) + \mathcal{O}(H_I^2)\}|0\rangle$$

free field result $= D_F(x-y)$

At order λ in $\lambda\varphi^4$ theory:

$$\langle 0|\, T\{\varphi(x)\varphi(y)(-i)\int dt \int d^3\vec{z}\, \frac{\lambda}{4!}\varphi^4(z)\}|0\rangle$$

$$= \langle 0|\, T\{\varphi(x)\varphi(y)\left(\frac{-i\lambda}{4!}\right)\int d^4z\, \varphi(z)\varphi(z)\varphi(z)\varphi(z)\}|0\rangle$$

Now apply Wick's Theorem

(Think how hard it would be without ...)

There is 1 term for each way of contracting 6 fields into pairs:

$5+4+3+2+1 = 15$

$$3 \quad \varphi(x)\varphi(y)\,\big(\varphi(z)\varphi(z)\varphi(z)\varphi(z) + 2 \text{ other}\big)$$

$+$

4 ways

$$12 \quad \varphi(x)\,\varphi(y)\,\varphi(z)\,\varphi(z)\,\varphi(z)\,\varphi(z)$$

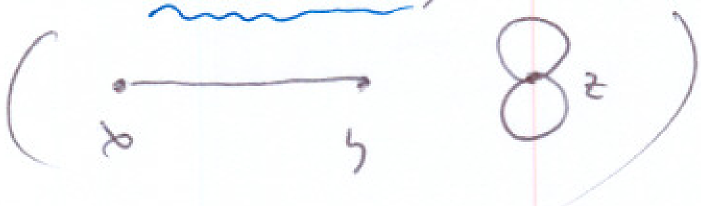
3 ways $\qquad$ 1 way

$$\therefore \quad \langle 0| T\{ \varphi(x)\,\varphi(y) \left(\frac{-i\lambda}{4!}\right)\int d^4z\, \varphi^4(z)\}|0\rangle$$

$$= 3\cdot\left(\frac{-i\lambda}{4!}\right) D_F(x-y)\int d^4z\, D_F(z-z)\,D_F(z-z)$$

$$+ 12\cdot\left(\frac{-i\lambda}{4!}\right)\int d^4z\, D_F(x-z)\,D_F(y-z)\,D_F(z-z)$$

= (Diagrammatically)



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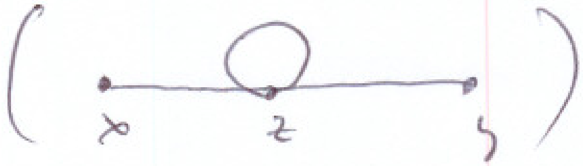


Lines are "propagators" dots are "vertices"

{Each "internal" point z has $(-i\lambda)\int d^4z$ associated with it}

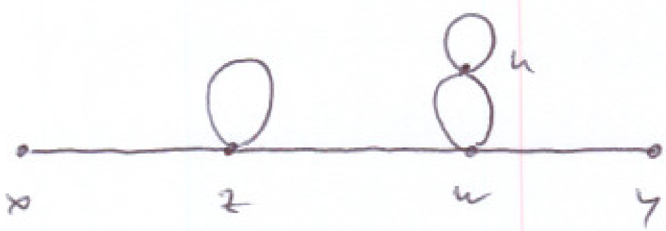
{Diagrams interpret algebraic expression as process of "creation", "propagation" and "destruction" in space time}

Consider more complicated example:

$\mathcal{O}(\lambda^3)$ (looking for simple rules!!)

$$\langle 0|\, \varphi(x)\varphi(y)\, \frac{1}{3!}\left(\frac{-i\lambda}{4!}\right)^3 \underbrace{\int d^4z\, \varphi\varphi\varphi\varphi}_{z}\, \underbrace{\int d^4w\, \varphi\varphi\varphi\varphi}_{w}\, \underbrace{\int d^4u\, \varphi\varphi\varphi\varphi}_{u}\, |0\rangle$$

$$= \frac{1}{3!}\left(\frac{-i\lambda}{4!}\right)^3 \int d^4z\, d^4w\, d^4u\, D_F(x-z)\, D_F(z-z)\, D_F(z-w)\, D_F(w-y)$$
$$D_F(w-u)\, D_F(w-u)\, D_F(u-u)$$

$=$ 

"CACTUS"

Combinatorics:

$$3! \;\times\; 4\cdot 3 \;\times\; 4\cdot 3\cdot 2 \;\times\; 4\cdot 3 \;\times\; \frac{1}{2}$$

interchange of vertices; placement of contractions into z vertex; " into w vertex; " into u vertex; interchange of w and u contractions

Total: 10,368 (1 diagram represents all of these!!)

(14 such operators, full contraction $\underline{\underline{135,135}}$)

Want simple rules to evaluate overall constant.

Note: (generically)

(A) $\frac{1}{n!}$ from Taylor series expansion

cancels $n!$ from interchanging vertices

(B) Generic vertex has 4 lines coming from 4 places: $4!$ (as in w vertex in example)

Cancels $\frac{1}{4!}$ from definition of coupling constant!!

What's left in "cactus" example if we include all factors

$$\boxed{\begin{aligned} &\frac{1}{3!}\left(\frac{1}{4!}\right)^3\left(3! \times \frac{4!}{2} \times 4! \times \frac{4!}{2} \times \frac{1}{2}\right) \\ &= \frac{1}{2\cdot 2\cdot 2} = \frac{1}{8} \end{aligned}}$$

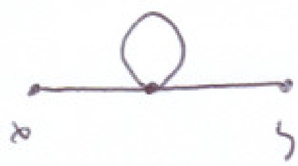
If have $n!$ from interchanging vertices and $4!$ for each vertex we get result that is too big by factor S: symmetry factor of diagram

{ in e.g. $S = 2\cdot 2\cdot 2 = 8$ }

How do we find S generally!? (factors of 2 ??)

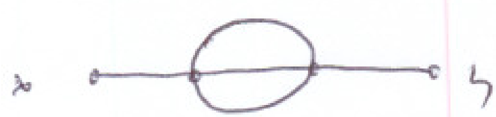
$$\boxed{\text{S = \# of ways of interchanging components without changing the diagram}}$$

Examples: S

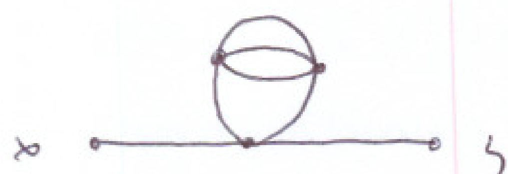


2

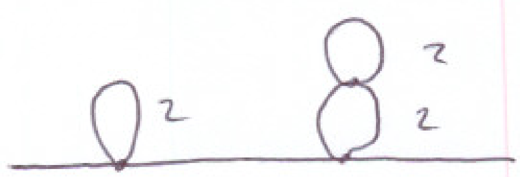
$2 \cdot 2 \cdot 2 = 8$



$3! = 6$



$3! \cdot 2 = 12$



$2 \cdot 2 \cdot 2 = 8$

In practice not really that relevant beyond $S = 2$.

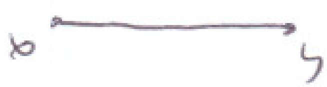
{ If you're not sure, you can count equivalent contractions }

Feynman Rules

Recall 2-pt function numerator

$$\boxed{\begin{aligned}&\langle 0|\, T\{\varphi_I(x)\,\varphi_I(y)\, e^{-i\int dt\, H_I(t)}\}\,|0\rangle \\ &= \text{sum of all possible diagrams (topologies) with 2 external points, built out of propagators and vertices.}\end{aligned}}$$

$\lambda\varphi^4$ Rules (coordinate space)

propagator  $= D_F(x-y)$

vertex  $= (-i\lambda)\int d^4z$

external point  $= 1$

combinatorics divide by S!

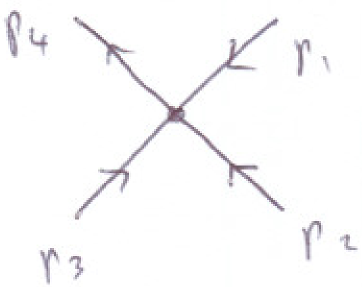
Notice that vertex contains interaction. Like Q.M. amplitude for emission + absorption of quanta.

$\int d^4z \Rightarrow$ sum over all points in space-time where process can occur.
(superposition)

Momentum Space

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\epsilon} e^{-ip\cdot(x-y)}$$ (Fourier Expansion)

Each propagator has 4-momentum p associated with it.
(arbitrary direction as $D_F(x-y) = D_F(y-x)$)



$$\int d^4z \, e^{-ip_1z - ip_2z - ip_3z + ip_4z}$$

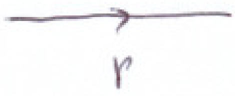
$$= (2\pi)^4 \delta^4(p_1 + p_2 + p_3 - p_4)$$

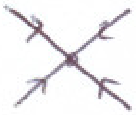
$\Rightarrow$ z-dependent factors give 4-momentum conservation δ-fn.

[Performing momentum integrals then gives Feynman Rules for momentum conserving processes.]

[i.e. no need to put δ-fns everywhere if momenta are labelled so as to conserve momenta!]

$\lambda\varphi^4$ Rules (momentum space)

propagator  $= \dfrac{i}{p^2 - m^2 + i\epsilon}$

vertex  $= -i\lambda$ (w/ momentum conservation at each vertex)

external point  $= e^{-ip\cdot x}$

combinatorics: divide by S!

undetermined momentum: $\displaystyle\int \frac{d^4p}{(2\pi)^4}$