Again:

$$\boxed{\begin{gathered} i\frac{\partial}{\partial t} U(t,t') = H_I(t)\, U(t,t') \\ U(t',t') = \mathbb{1} \end{gathered}}$$

From this equation one can work backwards and find:

$$\boxed{U(t,t') = e^{iH_0(t-t_0)}\, e^{-iH(t-t')}\, e^{-iH_0(t'-t_0)}}$$

$$\left\{ \text{Recall: } U(t,t_0) = e^{iH_0(t-t_0)}\, e^{-iH(t-t_0)} \right\}$$

$U(t,t')$ is unitary and satisfies:

(group identities)

$$U(t_1,t_2)\, U(t_2,t_3) = U(t_1,t_3)$$

$$U(t_1,t_3)\, U^{+}(t_2,t_3) = U(t_1,t_2)$$

We've found $\varphi(t,x_i)$ in terms of φ_I.

Now consider the interacting vacuum $|\Omega\rangle$

$|\Omega\rangle$ is the ground state of H

$|0\rangle$ is the ground state of H_0

So $$e^{-iHT}|0\rangle = \sum_n e^{-iHT}|n\rangle\langle n|0\rangle$$
$$= \sum_n e^{-iE_n}|n\rangle\langle n|0\rangle$$

$\{|n\rangle$ are eigenstates of H, $\sum_n |n\rangle\langle n| = \mathbb{1}\}$

As perturbation theory is valid, except that

$$\langle\Omega|0\rangle \neq 0$$

Then $|\Omega\rangle$ contributes in the sum over states:

$$e^{-iHT}|0\rangle = e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-iE_nT}|n\rangle\langle n|$$

$\{E_0 = \langle\Omega|H|\Omega\rangle$, $H_0|0\rangle = 0\}$

zero-point definition

As $E_n > E_0$ for all $n \neq 0$, we can get rid of excited states by waiting long enough!

Send $T \to \infty$ as $T \to \infty(1-i\epsilon)$

The ground state dies off most slowly.

$$\underset{T\to\infty(1-i\epsilon)}{e^{-iHT}|0\rangle} = e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle + \ldots$$

$$\Rightarrow \left(e^{-iE_0T}\langle\Omega|0\rangle\right)^{-1} e^{-iHT}|0\rangle = |\Omega\rangle + \ldots$$

or $$\boxed{|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_0T}\langle\Omega|0\rangle\right)^{-1} e^{-iHT}|0\rangle}$$

Now a dirty trick: as T is large,

$$T \to T + t_0$$

$t_0 \ll T$

$$\Rightarrow |\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_0(T+t_0)}\langle\Omega|0\rangle\right)^{-1} e^{-iH(T+t_0)}|0\rangle$$

$$= \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle\right)^{-1} e^{-iH(t_0-(-T))} \underbrace{e^{-iH_0(-T-t_0)}}_{1\text{ as } H_0|0\rangle=0}|0\rangle$$

Recall: $$U(t,t') = e^{iH_0(t-t_0)} e^{-iH(t-t')} e^{-iH_0(t'-t_0)}$$

$$\therefore\ U(t_0,-T) = e^{iH_0(t_0-t_0)} e^{-iH(t_0-(-T))} e^{-iH_0(-T-t_0)}$$

$$\Rightarrow U(t_0,-T) = e^{-iH(t_0-(-T))} e^{-iH_0(-T-t_0)}$$

So we have:

$$\boxed{|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle\right)^{-1} U(t_0,-T)|0\rangle}$$

That is,

$$|\Omega\rangle \sim U(t_0,-T)|0\rangle$$

time evolution from $-T$ to t_0 of $|0\rangle$ gives $|\Omega\rangle$!!

Similarly,

$$\boxed{\langle\Omega| = \lim_{T\to\infty(1-i\epsilon)} \langle 0|U(T,t_0)\left(e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle\right)^{-1}}$$

Now we can build: $\langle\Omega|\phi(x)\phi(y)|\Omega\rangle$

Assume partial time-ordering $x^0 > y^0 > t_0$

$$\left\{ \text{Recall: } \quad \phi = U^{\dagger}(t,t_0)\,\phi_I\,U(t,t_0) \right\}$$

$$\langle \Omega | \phi(x)\phi(y) | \Omega \rangle =$$

$$\left\{ \begin{array}{l} U(t_1,t_3)U^\dagger(t_2,t_3) = U(t_1,t_2) \\ U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3) \end{array} \right\}$$

$$\lim_{T\to\infty(1-i\epsilon)} \underbrace{\left(|\langle 0|\Omega\rangle|^2 e^{-iE_0(2T)} \right)^{-1}}_{\text{c-number prefactor}}$$

$$\times \langle 0| U(T,t_0) \left(U^\dagger(x^0,t_0)\, \phi_I(x)\, U(x^0,t_0) \right)$$

$$\times \left(U^\dagger(y^0,t_0)\, \phi_I(y)\, U(y^0,t_0) \right) U(t_0,-T) |0\rangle$$

$$= \lim_{T\to\infty(1-i\epsilon)} \;\; '' \;\; \times \langle 0| U(T,x^0)\, \phi_I(x)\, U(x^0,y^0)\, \phi_I(y)\, U(y_0,-T)|0\rangle$$

{where we used "group identities" of U}

we can get rid of pesky c-number prefactor by normalization!

$$1 = \langle \Omega | \Omega \rangle = \left(|\langle 0|\Omega\rangle|^2 e^{-iE_0(2T)} \right)^{-1}$$

$$\times \langle 0| U(T,t_0)\, U(t_0,-T)|0\rangle$$

So for $x^0 > y^0 > t_0$:

$$\boxed{\langle \Omega | \phi(x)\phi(y) | \Omega \rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0| U(T,x^0)\, \phi_I(x)\, U(x^0,y^0)\, \phi_I(y)\, U(y_0,-T)|0\rangle}{\langle 0| U(T,-T)|0\rangle}}$$

Note that all fields are in time order!!

For $y^0 > x^0$ this is still true!

Thus,

$$\boxed{\langle\Omega|T\{\varphi(x)\varphi(y)\}|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\{\varphi_I(x)\varphi_I(y)e^{-i\int_{-T}^{T}dt\, H_I(t)}\}|0\rangle}{\langle 0|T\{e^{-i\int_{-T}^{T}dt\, H_I(t)}\}|0\rangle}}$$

- Entirely written in terms of φ_I for which we have explicit expression in terms of ladder operators
- $T\{\}$ product makes final result simple.
- Readily generalized to products of arbitrary numbers of fields.
- Useful for perturbation expansions.

E.g. $H_I = \int d^3x \frac{\lambda}{4!}\varphi_I^4$