Ⓑ Here we will focus on $\mathcal{H}_{int} = -\mathcal{L}_{int}$ that are functions of fields only, <u>not derivatives</u>.

(many interesting systems don't satisfy this ...)

EXAMPLE A

$$\boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4}$$

(lambda phi fourth theory)

Consider <u>dimensional</u> analysis

$\Rightarrow \quad \varphi \sim [m] \sim [\ell]^{-1}$

$\therefore$ λ is dimensionless coupling. (like e)

{ Note that could consider $\mathcal{L}_{int} = -\frac{\lambda}{3!}\varphi^3$
But $\mathcal{H}_{int}$ is then unbounded from below }

$\lambda\varphi^4$ theory is nice pedagogical example.

Also relevant to the 'real world':

Higgs interactions

Landau-Ginzburg theory of superconductivity.

Let's try to solve it!

Equations of motion:

$$\Box\varphi = \frac{\partial\mathcal{L}}{\partial\varphi} = -m^2\varphi - \frac{\lambda}{3!}\varphi^3$$

$$\Rightarrow \boxed{(\Box + m^2)\varphi = -\frac{\lambda}{3!}\varphi^3}$$

This is highly <u>non-linear</u> equation:

Cannot be solved by Fourier analysis

In quantizing, we impose

$$[\varphi(x_i), \pi(y_i)] = i\,\delta^{(3)}(x_i - y_i)$$

$\frac{\partial \mathcal{L}}{\partial \dot\varphi}$ definition would change if $\mathcal{L}$ contained derivatives.

Unaffected by interaction.

EXAMPLE B Quantum Electrodynamics

$$\mathcal{L}_{QED} = \mathcal{L}_{DIRAC} + \mathcal{L}_{MAXWELL} + \mathcal{L}_{int}$$

$$\boxed{\mathcal{L}_{QED} = \bar\Psi_e (i\not\partial - m_e)\Psi_e - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} - e\bar\Psi_e \gamma^\mu \Psi_e A_\mu}$$

A_μ is E+M field (photon)

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

(field strength tensor)

$e = -|e|$ is electron charge

D.A.:

$$\int d^4x \; \bar{\psi}\partial\psi \sim [m]^0$$

$$\Rightarrow \bar{\psi}\partial\psi \sim [m]^4 \quad \Rightarrow \bar{\psi}\psi \sim [m]^3$$

$$\Rightarrow \psi \sim [m]^{3/2} \sim [\ell]^{-3/2}$$

$A_\mu \sim [m]$ $\quad e \sim [m]^0$ dimensionless

Evidently this simple $\mathcal{L}_{QED}$ accounts for most observable phenomena down to 10^{-13}cm!!

More compactly,

$$\boxed{\mathcal{L}_{QED} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}}$$

$D_\mu \equiv \partial_\mu + ieA_\mu$ "gauge covariant derivative"

$\mathcal{L}_{QED}$ has invariance under:

$$\psi(x) \to e^{i\alpha(x)}\psi(x)$$

$$A_\mu(x) \to A_\mu - \frac{1}{e}\partial_\mu\alpha(x)$$

U(1) gauge symmetry (deep geometrical significance)

Checks:

$$\bar{\psi} i \not\partial \psi \to \bar{\psi} e^{-i\alpha(x)} i (+i \not\partial \alpha(x)) e^{i\alpha(x)} \psi$$

$$+ \bar{\psi} e^{-i\alpha(x)} i e^{i\alpha(x)} \not\partial \psi$$

$$= \bar{\psi} \gamma^\mu (i\partial_\mu - (\partial_\mu \alpha(x))) \psi$$

$$-e \bar{\psi} \not{A} \psi \to -e \bar{\psi} \gamma^\mu (A_\mu - \tfrac{1}{e}(\partial_\mu \alpha(x))) \psi$$

$$= -e \bar{\psi} \not{A} \psi + \bar{\psi} \gamma^\mu (\partial_\mu \alpha(x)) \psi$$

$$\Rightarrow \quad \bar{\psi} i \not{D} \psi \to \bar{\psi} i \not{D} \psi$$

INVARIANT under $U(1)_{EM}$

($F_{\mu\nu} F^{\mu\nu}$ invariance is trivial)

Equations of Motion?

ψ: $\quad \boxed{(i\not{D} - m)\psi(x) = 0}$

A_μ: $\quad \boxed{\partial_\mu F^{\mu\nu} = e \bar{\psi} \gamma^\nu \psi = e j^\nu}$

(inhomogeneous Maxwell Eq's)

{ Quantization of A_μ is tricky using $\mathcal{H}$.
Better to use path integral approach }

EXAMPLE C Yukawa Theory

$$\boxed{\mathcal{L}_Y = \bar{\psi} i \partial\!\!\!/ \psi + \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2} m^2 \varphi^2 - g \bar{\psi}\psi\varphi}$$

$\left\{ \text{D.A.} \quad \int d^4x\, g\, \bar{\psi}\psi\varphi \sim (m)^0 \quad g \sim (m)^0 \ \text{dimensionless} \right\}$

Yukawa interactions are extremely important in the Standard Model!

What is most general list of interactions for our model Lagrangians?

(A) $\mathcal{L}' = a\,\varphi^6 + b\,(\partial_\mu \varphi)^2 (\partial_\nu \varphi)^2 \varphi^3 + \ldots$

(B) $\mathcal{L}' = \alpha\,(F_{\mu\nu} F^{\mu\nu})^2 + \beta\,(\bar{\psi}\psi)^2 + \ldots$

(C) $\mathcal{L}' = \gamma\,(\varphi \bar{\psi}\psi)^2 + \delta\, \bar{\psi}\, \partial_\mu \varphi\, \psi \gamma^\mu \varphi^3 + \ldots$

∞ # of operators consistent w/ symmetries!!

One simple principle eliminates all these interactions!!

$\boxed{\text{RENORMALIZABILITY}}$

Notice <u>dimensionality</u> of operators:

$$\boxed{a \to \frac{\bar a}{\Lambda^2} \qquad b \to \frac{\bar b}{\Lambda^4} \qquad \alpha \to \frac{\bar\alpha}{\Lambda^4} \qquad \beta \to \frac{\bar\beta}{\Lambda^2} \qquad \gamma \to \frac{\bar\gamma}{\Lambda^4} \qquad \delta \to \frac{\bar\delta}{\Lambda^4}}$$

(barred couplings are dimensionless)

Interactions with negative mass dimension couplings are <u>NON-RENORMALIZABLE</u>

<u>Example:</u> <u>$\lambda\varphi^4$</u>

divergent $k \to \infty$

Divergences can be absorbed into <u>redefinition</u> of m^2 and λ.

True only for <u>RENORMALIZABLE</u> interactions

RENORMALIZABLE INTERACTIONS

(objects of dim ~ $[m]^4$ or less)

(A) $\mu\varphi^3$, $\lambda\varphi^4$

$\mu \sim [m]$ "SUPER" RENORMALIZABLE

(B) $e\bar{\psi}\gamma^\mu\psi A_\mu$

$(A_\mu A^\mu)^2$, etc. break $U(1)_{em}$

(C) $g\bar{\psi}\psi\varphi$

The Standard Model of particle interactions is built out of these types of interactions!!

Notice how constraining Lorentz invariance, symmetries, renormalizability, etc. are on the interactions of QFT.

(Compare w/ N.R. Q.M.)

How do we account for these interactions??

No exactly solvable interacting QFT is known in more than 1+1 spacetime dimensions!!

NEED $\boxed{\text{PERTURBATION THEORY}}$

For perturbation theory to be useful,

$\underline{H_{int} \text{ must be small}}$

(i.e. dimensionless coupling constants must be in some sense small)

We will derive spacetime picture of perturbation theory from conventional technology of Q.M. (pioneered by Dyson)

Focus on calculating:

$$\boxed{\langle \Omega | \, T\{ \phi(x) \phi(y) \} \, | \Omega \rangle} \quad (*)$$

$|\Omega\rangle \neq |0\rangle$ is the <u>interacting</u> vacuum.

{ Recall: this <u>correlation function</u> is the amplitude for propagation of particle between y and x. }

In <u>FREE</u> field theory:

$$\langle 0 | \, T\{ \phi(x) \phi(y) \} \, | 0 \rangle = D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} \, e^{-ip\cdot(x-y)}$$

How does this object change when interaction is turned on??

$$\boxed{H = H_0 + H_{int} = H_{K.G.} + \int d^3x \, \frac{\lambda}{4!} \varphi^4(x_i)}$$

Schrödinger picture field

{ Recall:

$$H_{K.G.} = \int d^3x \left[\tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\nabla_i \varphi)^2 + \tfrac{1}{2} m^2 \varphi^2 \right]$$ }

H_{int} enters ⊛ in two places:

1. $|\Omega\rangle$ (interacting vacuum state)

2. $$\varphi(x) = e^{iHt} \varphi(x_i) e^{-iHt}$$

Heisenberg picture field

Includes H_{int}

At some fixed time t_0:

$$\varphi(t_0, x_i) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_{p_i} e^{ip_i x_i} + a_{p_i}^\dagger e^{-ip_i x_i} \right)$$

(This defines the Schrödinger picture field at reference time t_0.)

How do we obtain $\varphi(t, x_i)$ for $t \neq t_0$?

Formally, we switch to Heisenberg picture:

$$\varphi(t, x_i) = e^{iH(t-t_0)} \varphi(t_0, x_i)\, e^{-iH(t-t_0)}$$

If $\lambda \to 0$, $H \to H_0$

$$\Rightarrow \varphi(t, x_i)\Big|_{\lambda=0} = e^{iH_0(t-t_0)} \varphi(t_0, x_i)\, e^{-iH_0(t-t_0)}$$

$$\equiv \boxed{\varphi_I(t, x_i)}$$

"Interaction picture field"

When λ is small, φ_I will give most important part of the time dependence of $\varphi(t, x_i)$.

(OR ELSE perturbation theory is not valid!)

Explicitly:

$$\varphi_I(t, x_i) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_{p_i} e^{-ip\cdot x} + a_{p_i}^{\dagger} e^{ip\cdot x} \right)\Big|_{x^0 = t - t_0}$$

(FREE FIELD EXPRESSION)

Inversely $\Rightarrow$

$$\varphi(t_0, x_i) = e^{-iH_0(t-t_0)} \varphi_I(t, x_i)\, e^{iH_0(t-t_0)}$$

$\therefore$

$$\varphi(t, x_i) = e^{iH(t-t_0)} e^{-iH_0(t-t_0)} \varphi_I(t, x_i) e^{iH_0(t-t_0)} e^{-iH(t-t_0)}$$

$$\Rightarrow \quad \underline{\varphi(t, x_i) = U^\dagger(t, t_0)\, \varphi_I(t, x_i)\, U(t, t_0)}$$

$$\boxed{U(t, t_0) = e^{iH_0(t-t_0)} e^{-iH(t-t_0)}}$$

(Unitary) interaction picture propagator

GOAL: Express $U(t, t_0)$ in terms of φ_I for which we have expansion in ladder operators.

$U(t, t_0)$ satisfies boundary condition

$$\boxed{U(t_0, t_0) = \mathbb{1}}$$

Differentiating $\Rightarrow$

$$i \frac{\partial}{\partial t} U(t, t_0) = e^{iH_0(t-t_0)} (H - H_0) e^{-iH(t-t_0)}$$

$$= e^{iH_0(t-t_0)} H_{int}\, e^{-iH(t-t_0)}$$

$$= \underbrace{e^{iH_0(t-t_0)} H_{int}\, e^{-iH_0(t-t_0)}}_{\displaystyle \shortparallel \atop \displaystyle \int d^3x \frac{\lambda}{4!} \varphi_I^4} \underbrace{e^{iH_0(t-t_0)} e^{-iH(t-t_0)}}_{\displaystyle \shortparallel \atop \displaystyle U(t, t_0)}$$

$$\Rightarrow \boxed{i\frac{\partial}{\partial t} U(t,t_0) = H_I(t)\, U(t,t_0)}$$

$$H_I(t) = \int d^3x \, \frac{\lambda}{4!} \phi_I^{\,4} \quad \left(= e^{iH_0(t-t_0)} H_{int}\, e^{-iH_0(t-t_0)} \right)$$

interaction Hamiltonian in the interaction picture

Solution ??

$$U \sim e^{-iH_I t}$$

Actual Solution: power series in λ !!

$$\boxed{\begin{aligned} U(t,t_0) = \mathbb{1} &+ (-i)\int_{t_0}^{t} dt_1\, H_I(t_1) \\ &+ (-i)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_I(t_1)\, H_I(t_2) \\ &+ \dots \end{aligned}}$$

Straightforward to verify that this is solution w/ correct boundary condition.

$\left\{ \text{Note: Solution well adapted to perturbation theory.} \right\}$

Notice that terms are time ordered:

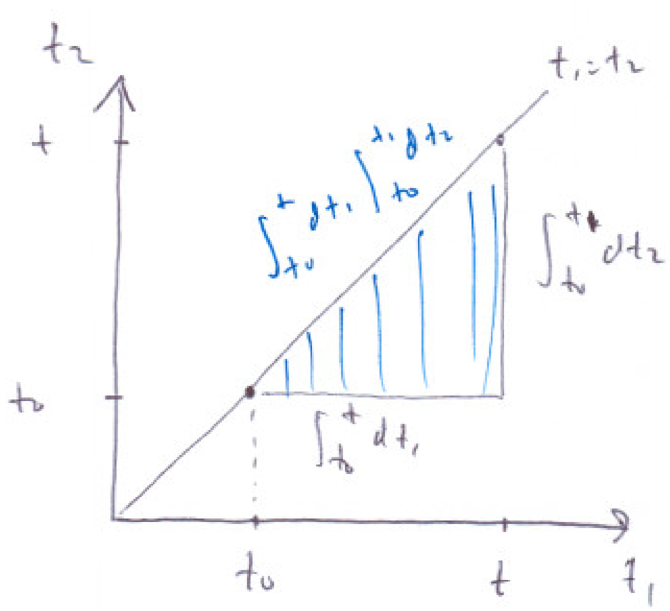
$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \; H_I(t_1)\, H_I(t_2)$$

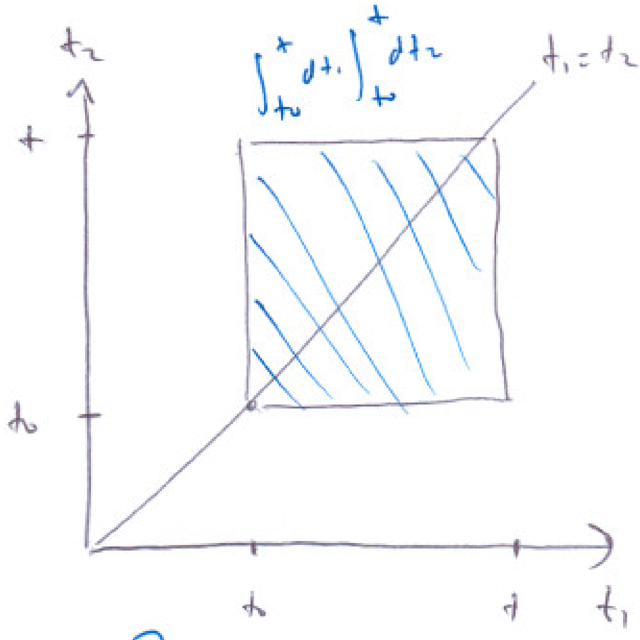
$$= \frac{1}{2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \; T\{ H_I(t_1)\, H_I(t_2) \}$$

CHECK:

$$\frac{1}{2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \Big(\theta(t_1 - t_2)\, H_I(t_1)\, H_I(t_2) + \theta(t_2 - t_1)\, H_I(t_2)\, H_I(t_1) \Big)$$

Geometric interpretation





$T\{ H_I(t_1)\, H_I(t_2) \}$ is symmetric about $t_1 = t_2$

Similarly, higher order terms can be simplified w/ the time ordering symbol.

$$\Rightarrow \boxed{U(t,t_0) = T\left\{ e^{-i\int_{t_0}^{t} dt'\, H_I(t')} \right\}}$$

{ Here $T\{u\}$ means that applies to each term in Taylor series expansion }

Note that

$$\boxed{\varphi(t,x_i) = T\left\{ e^{-i\int_{t_0}^{t} dt' \frac{\lambda}{4!} \varphi_I^4} \right\} \varphi_I(t,x_i)\, T\left\{ e^{i\int_{t_0}^{t} dt' \frac{\lambda}{4!} \varphi_I^4} \right\}}$$

$\varphi(t,x_i)$ is given entirely in terms of φ_I !!

We can generalize U:

$$U(t,t') = T\left\{ e^{-i\int_{t'}^{t} dt''\, H_I(t'')} \right\} \quad t \geq t'$$

{ t' is time other than reference time t_0 }

Satisfies same differential equation:

$$i\frac{\partial}{\partial t} U(t,t') = H_I(t)\, U(t,t')$$