For logical development, see Weinberg

Read historical intro. in Weinberg VI.

$$\boxed{\text{QM} + \text{Relativity} \Rightarrow \text{Q.F.T}}$$

$$\hbar = c = 1$$

$$(\text{length}) = (\text{time}) = (\text{energy})^{-1} = (\text{mass})^{-1}$$

high - energy = short distance

Tensors + Relativity

Our stage is space - time

Minkowski Space

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & 0 \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

<u>Greek</u> indices run over $0, 1, 2, 3$
t, x, y, z

<u>i.e.</u> $\mu, \nu, \alpha, \beta,$ <u>etc.</u>

<u>Roman</u> indices run over $1, 2, 3$
x, y, z

<u>i.e.</u> $i, j, k,$ <u>etc.</u>

$$X^\mu = (X^0, X^i) \quad \left(\text{or } (x^0, \vec{x}) \right)$$

"<u>Contravariant</u>"

$$x_\mu = g_{\mu\nu} X^\nu = (x^0, -x^i)$$

"<u>Covariant</u>"

$$p \cdot x = p^\mu x_\mu = g_{\mu\nu} p^\mu x^\nu = p^0 x^0 - p^i x^i$$

$$\left(\text{or } p^0 x^0 - \vec{p} \cdot \vec{x} \right)$$

Massive particle dispersion relation:

$$p^2 = p \cdot p = p^\mu p_\mu = E^2 - \vec{p}^{\,2} = m^2$$

$$E = \sqrt{\vec{p}^{\,2} + m^2}$$

x^μ is "naturally" contravariant.

whereas

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \nabla_i \right) = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^i} \right)$$

is "naturally" covariant.

Recall the 3-d antisymmetric tensor:

ϵ_{ijk} "Levi-Civita symbol"

$$\left(\begin{array}{l} \epsilon^{123} = 1 \\ \epsilon^{213} = -1 \end{array} \quad \text{etc.} \right)$$

In Minkowski space we have 4-d antisymmetric tensor:

$$\epsilon^{\mu\nu\rho\sigma} \qquad \left(\begin{array}{l} \epsilon^{0123} = +1 \\ \epsilon^{1230} = -1 \end{array} \quad \text{etc.} \right)$$

Quantum Mechanics

$$\hat{H}\ (\sim E) = i\frac{\partial}{\partial x^0} \qquad \hat{p}_i = -i\nabla_i$$

$$p^\mu = i\partial^\mu$$

$$\psi(x) = \psi(0)\, e^{-ik\cdot x}$$

$$i\partial^\mu[\psi(x)] = i(-ik^\mu)\,\psi(x) = k^\mu\,\psi(x)$$

Distributions

$$\Theta(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

(Heavyside Step fn.)

Integral representation:

$$\Theta(x) = \lim_{\epsilon \to 0} \left(-\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-ix\tau}}{\tau + i\epsilon} \, d\tau \right)$$

$$\frac{d}{dx}\Theta(x) = \delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\tau} \, d\tau$$

Dirac delta fn.

Generalization to n-dimensions:

$$\int d^n x \, \delta^{(n)}(x) = 1$$

4-d $\quad \underbrace{\delta^{(4)}(k) = \frac{1}{(2\pi)^4} \int d^4 x \, e^{ikx}}$

$$\left(\int d^4 x = \int dx^0 dx^1 dx^2 dx^3 \right)$$

Electrodynamics

(Heaviside - Lorentz conventions)

$$\alpha = \frac{e^2}{4\pi} = \frac{e^2}{4\pi \hbar c} \simeq \frac{1}{137}$$

e: charge of the electron (negative)

Maxwell Equations

$$\epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = 0 \;\rightarrow\; \boxed{\mathrm{div}\, B_i = 0}, \quad \boxed{\mathrm{curl}\, E_i + \frac{\partial}{\partial t} B_i = 0}$$

$$\partial_\mu F^{\mu\nu} = e j^\nu \;\rightarrow\; \boxed{\mathrm{div}\, E_i = \rho}, \quad \boxed{\mathrm{curl}\, B_i - \frac{\partial E_i}{\partial t} = j_i}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$A^\mu = (\Phi, A^i) \qquad j^\nu = (\rho, j^i)$$

In next few weeks there will be lots of formalism. Consider simple example of what we are after.

Most commonly calculated object in QFT:

<u>scattering cross section</u> σ

$$\frac{d\sigma}{d\Omega} = \frac{\#\text{ particles detected / second}}{\text{initial flux of particles}}$$

{ flux = # of particles passing through a unit area transverse to the beam / second }

Simplest example in Q.E.D.

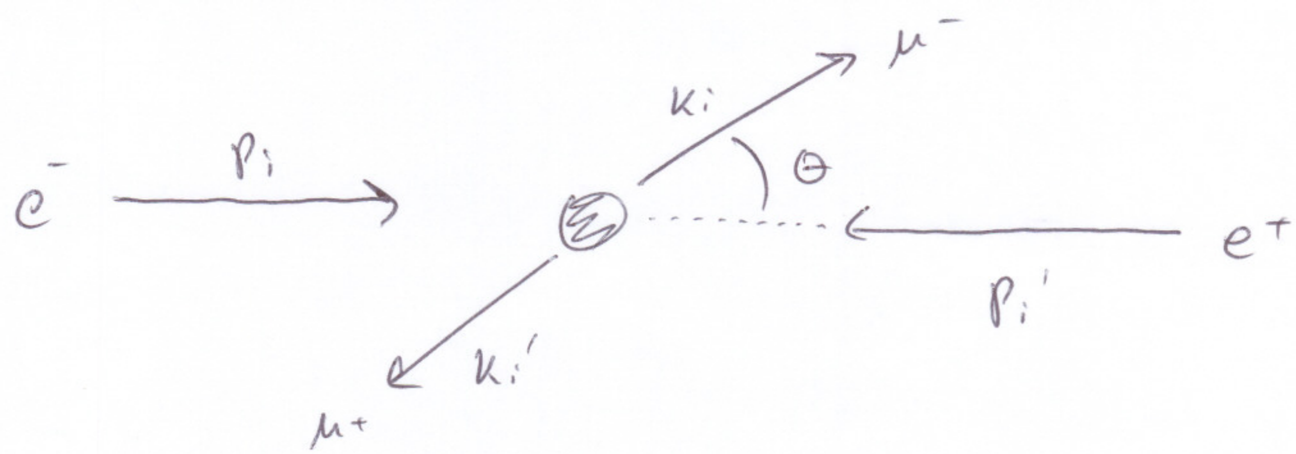
$$e^+ e^- \rightarrow \mu^+ \mu^-$$

positron electron antimuon muon

Antiparticles are predictions of Q.F.T.
(as we will see)

Choose special Lorentz frame:

center-of-mass



$$p_i + p_i' = 0$$

$$k_i + k_i' = 0$$

Conservation of energy?

$$E_{e^-} = \sqrt{\vec{p}^2 + m_e^2} = E_{e^+}$$

$$E_{\mu^-} = \sqrt{\vec{k}^2 + m_\mu^2} = E_{\mu^+}$$

If $|\vec{p}|, |\vec{k}| \gg m_e, m_\mu$

$$|\vec{p}| = |\vec{k}| = E \equiv \frac{E_{cm}}{2} = |\vec{p}'| = |\vec{k}'|$$

$$\vec{k} \cdot \vec{p}' = \vec{k}' \cdot \vec{p} = |\vec{p}|^2 \cos\theta = E^2 \cos\theta = \frac{E_{cm}^2}{4} \cos\theta$$

e and μ have spin $\frac{1}{2}$ $\uparrow$ $\downarrow$

Usually beams are unpolarized:

average over initial spins

sum over final spins

If polarized must specify spin orientation relative to some axis (e.g. direction of motion)

For given spin orientations:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 E_{cm}^2} |\mathcal{M}|^2}$$

Differential cross section w/ μ^- produced into a solid angle $d\Omega$.

Follows from dimensional analysis

massless particles $\Rightarrow$ 1 scale : E_{cm}

$$\sigma \sim \text{area} \sim \text{length}^2 \sim E_{cm}^{-2}$$

M is Q.M. amplitude for process to occur.

Main purpose of this course is to learn how to compute M !!

Almost never have exact expression for M.

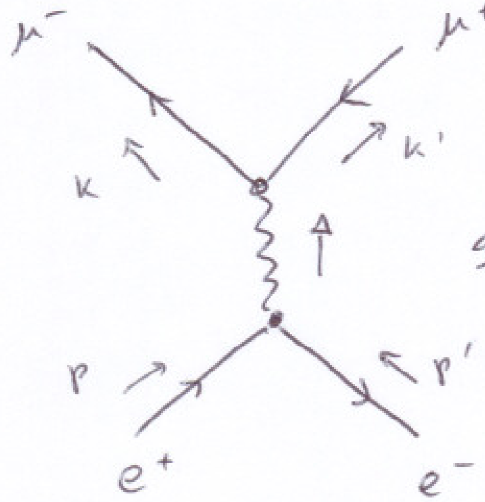
Rely on perturbation theory : here in $\alpha \sim \frac{1}{137}$

Feynman came up with nice method to handle algebraic mess.

Feynman Rules

Draw all topologies:

$e^+ e^- \to \mu^+ \mu^-$



$q = p + p' = k + k'$

Each "diagram" contributes to $\mathcal{M}$.

Non-trivial !!

In Q.M. perturbation theory,

1st order: $\langle \text{final} \mid H_I \mid \text{initial} \rangle$

Interaction part of Hamiltonian

In our example:

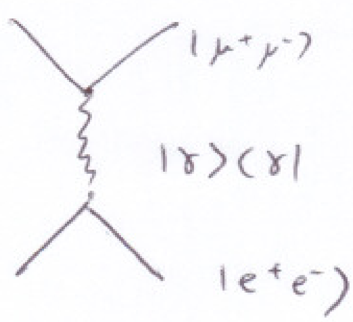
$$\mathcal{M} \sim \langle \mu^+ \mu^- | H_I | e^+ e^- \rangle$$

But initial and final states couple only through E & M interaction!!

Must go to $2^{\underline{nd}}$ order:

$$\mathcal{M} \sim \langle \mu^+ \mu^- | H_I | \gamma \rangle_\mu \langle \gamma | H_I | e^+ e^- \rangle^\mu$$

Each part must be Lorentz 4-vector.



Angular momentum considerations $\Rightarrow$

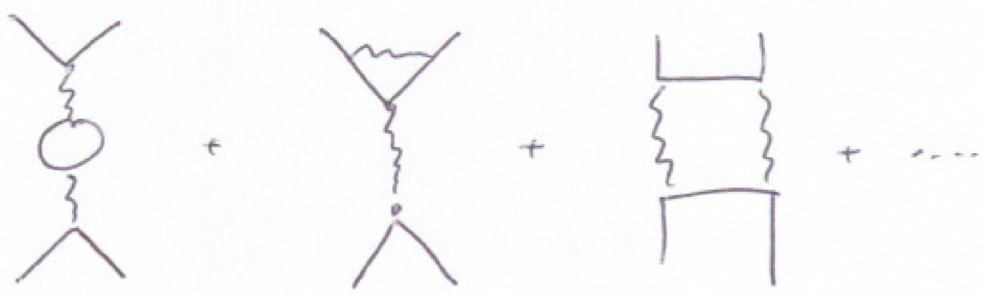
$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4 E_{c.m.}^2} (1 + \cos^2\theta)}$$

$$\boxed{\sigma_{tot} = \frac{4\pi\alpha^2}{3E_{cm}^2}}$$

agrees within 10% of experiment

Next order in perturbation theory ??

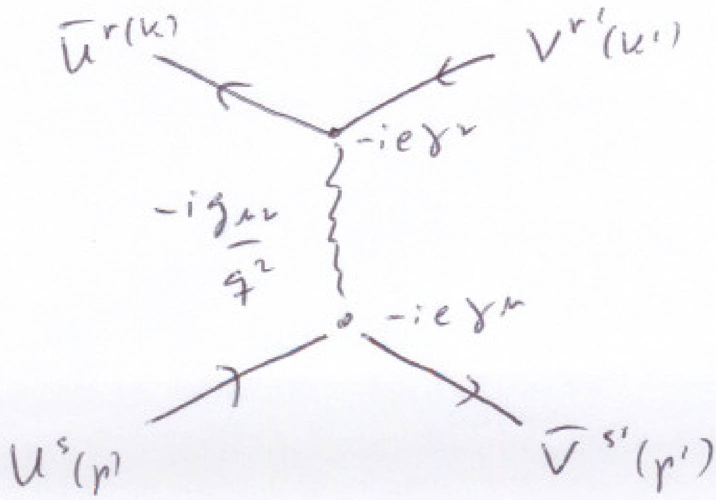
$\mathcal{O}(\alpha^3)$ in cross section



Notice the loops !!

$\sim \int d^4k$

Feynman Rules



Put it all together:

$$\mathcal{M} = \bar{v}^{s'}(p')(-ie\gamma^\mu)u^s(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}^r(k)(-ie\gamma^\nu)v^{r'}(k')$$

$$= \frac{ie^2}{q^2}\left[\bar{v}^{s'}(p')\gamma^\mu u^s(p)\right]\left[\bar{u}^r(k)\gamma_\mu v^{r'}(k')\right]$$

We will learn how to deal with this!

Necessity of Fields

Why fields and not particles?

Generalize Q.M. wave equation to be consistent with relativity:

K-G equation, Dirac equation

We will see that this gives rise to inconsistencies

There is intuitive way to see necessity of field viewpoint:

Consider relativistic dispersion relation for particle of mass M.

$$E = \sqrt{p^2 + M^2} = M\left(1 + \frac{p^2}{2M^2} + \mathcal{O}(p^4)\right)$$

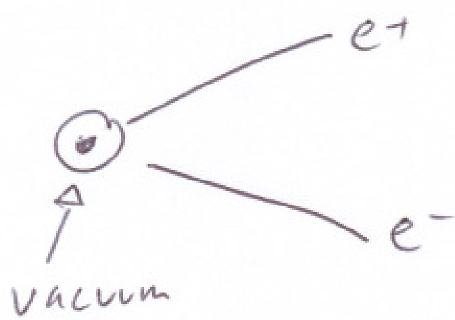
for $p \ll M$

$$E = \frac{p^2}{2m} \rightarrow \frac{\nabla^2}{2m} \quad \text{etc.}$$

But for $p \sim M$ can pair produce!!

$$E = mc^2$$

(mass out of energy)

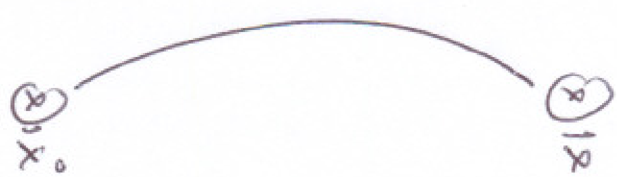


pair creation

Can have complicated vacuum with indeterminate # of particles.

CAUSALITY

Consider free particle propagation:



QM amplitude??

$$U(t) = \langle \vec{x} | \, e^{-i\hat{H}t} \, | \vec{x}_0 \rangle$$

$$= \langle \vec{x} | e^{-i\hat{\vec{p}}^2/2m\, t} | \vec{x}_0 \rangle$$

$$= \int \frac{d^3p}{(2\pi)^3} \langle \vec{x} | e^{-i\hat{\vec{p}}^2/2m\, t} | \vec{p} \rangle \langle \vec{p} | \vec{x}_0 \rangle$$

$$= \int \frac{d^3p}{(2\pi)^3} e^{-i\vec{p}^2/2m\, t} e^{i\vec{p}\cdot(\vec{x}-\vec{x}_0)}$$

$$\Rightarrow \boxed{U(t) = \left(\frac{m}{2\pi i t}\right)^{3/2} e^{im(\vec{x}-\vec{x}_0)^2/2t}}$$

$U(t)$ is non-zero for all $\vec{x}$ and t !!

Therefore, a particle can propagate between any two points in an arbitrarily short time.

$\boxed{\text{Relativity} \Rightarrow \text{signals bounded by } c\,!}$

$\therefore$ This expression violates causality.

What if we use relativistic dispersion relation instead?

$$E = \sqrt{\vec{p}^2 + m^2}$$

$$U(t) = \langle \vec{x} | \, e^{-it\sqrt{\hat{\vec{p}}^2 + \hat{m}^2}} \, | \vec{x}_0 \rangle$$

$$= \frac{1}{(2\pi)^3} \int d^3p \; e^{-it\sqrt{\vec{p}^2 + m^2}} \; e^{i\vec{p}\cdot(\vec{x} - \vec{x}_0)}$$

One finds:

$$\boxed{U(t) \sim e^{-m\sqrt{x^2 - t^2}}}$$

Propagation is small but non-zero outside the light cone $\Rightarrow$ Causality still violated!!

How does Q.F.T solve this problem?