solutions from http://bolvan.ph.utexas.edu/ vadim

<u>Textbook Problem **4.3**(a):</u>

The propagators are contractions of the free fields, thus for N distinct fields Φ^i of the same mass m we have

$$\Phi^j(x)\,\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet\,\Phi^k(y) \;=\; \contraction{}{\Phi}{^j(x)\,}{\Phi}\Phi^j(x)\,\Phi^k(y) \;=\; \delta^{jk}\,G^F(x-y)_{\mathrm{mass}=m}\,, \tag{1}$$

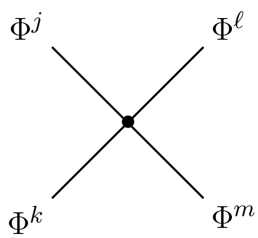
or in momentum space,

$$\Phi^j \;\overline{}\; \Phi^k \;=\; \frac{i\delta^{jk}}{q^2-m^2+i0}\,. \tag{2}$$

The vertices follow from the perturbation operator

$$\hat{V} \;=\; \int d^3\mathbf{x}\left(\tfrac{\lambda}{4}(\Phi\cdot\Phi)^2 \;\equiv \sum_j \tfrac{\lambda}{4}(\hat{\Phi}^j)^4 \;+\sum_{j<k}\tfrac{\lambda}{2}(\hat{\Phi}^j)^2(\hat{\Phi}^k)^2\right), \tag{3}$$

hence two vertex types: (1) a vertex involving 4 lines of the same field species Φ^j, with the vertex factor $-i\frac{\lambda}{4}\times 4! = -6i\lambda$; and (2) a vertex involving 2 lines of one field species Φ^j and 2 lines of a different species Φ^k, with the vertex factor $-i\frac{\lambda}{2}\times(2!)^2 = -2i\lambda$. (The combinatoric factors arise from the interchanges of the identical fields in the same vertex, thus 4! for the first vertex type and $(2!)^2$ for the second type.) Equivalently, we may use a single vertex type involving 4 fields of whatever species, with the species-dependent vertex factor



$$= \; -2i\lambda\bigl(\delta^{jk}\delta^{\ell m}+\delta^{j\ell}\delta^{km}+\delta^{jm}\delta^{k\ell}\bigr). \tag{4}$$

Now consider the scattering process $\Phi^j+\Phi^k\to\Phi^\ell+\Phi^m$. At the lowest order of the perturbation theory, there is just one Feynman diagram for this process; it has one vertex, 4 external legs and no internal lines. Consequently, at the lowest order,

$$\mathcal{M}(\Phi^j+\Phi^k\to\Phi^\ell+\Phi^m) \;=\; -2\lambda\bigl(\delta^{jk}\delta^{\ell m}+\delta^{j\ell}\delta^{km}+\delta^{jm}\delta^{k\ell}\bigr) \tag{5}$$

independent of the particles' momenta. Specifically,

$$\begin{aligned}
\mathcal{M}(\Phi^1+\Phi^2\to\Phi^1+\Phi^2) &= -2\lambda,\\
\mathcal{M}(\Phi^1+\Phi^1\to\Phi^2+\Phi^2) &= -2\lambda,\\
\mathcal{M}(\Phi^1+\Phi^1\to\Phi^1+\Phi^1) &= -6\lambda,
\end{aligned} \tag{6}$$

and consequently (using eq. (4.85) of the textbook)

$$\begin{aligned}
\frac{d\sigma(\Phi^1+\Phi^2\to\Phi^1+\Phi^2)}{d\Omega_{\text{c.m.}}} &= \frac{\lambda^2}{16\pi^2E^2_{\text{c.m.}}}\,,\\
\frac{d\sigma(\Phi^1+\Phi^1\to\Phi^2+\Phi^2)}{d\Omega_{\text{c.m.}}} &= \frac{\lambda^2}{16\pi^2E^2_{\text{c.m.}}}\,,\\
\frac{d\sigma(\Phi^1+\Phi^1\to\Phi^1+\Phi^1)}{d\Omega_{\text{c.m.}}} &= \frac{9\lambda^2}{16\pi^2E^2_{\text{c.m.}}}\,.
\end{aligned}\tag{7}$$

These are *partial* cross sections. To calculate the total cross sections, we integrate over $d\Omega$, which gives the factor of 4π when the two final particles are of distinct species, but for the same species, we only get 2π because of Bose statistics. Hence,

$$\begin{aligned}
\sigma_{\text{tot}}(\Phi^1+\Phi^2\to\Phi^1+\Phi^2) &= \frac{\lambda^2}{4\pi E^2_{\text{c.m.}}}\,,\\
\sigma_{\text{tot}}(\Phi^1+\Phi^1\to\Phi^2+\Phi^2) &= \frac{\lambda^2}{8\pi E^2_{\text{c.m.}}}\,,\\
\sigma_{\text{tot}}(\Phi^1+\Phi^1\to\Phi^1+\Phi^1) &= \frac{9\lambda^2}{8\pi E^2_{\text{c.m.}}}\,.
\end{aligned}\tag{8}$$

Textbook Problem **4.3**(b):

The classical potential

$$V(\Phi^2) \;=\; -\tfrac{1}{2}\mu^2(\Phi^2) \;+\; \tfrac{1}{4}\Lambda(\Phi^2)^2 \tag{9}$$

with a negative mass term $m^2=-\mu^2<0$ has a minimum (or rather a spherical shell of minima) for

$$\Phi^2\equiv\Phi\cdot\Phi \;=\; v^2 \;=\; \frac{\mu^2}{\lambda} \;>\; 0. \tag{10}$$

Semi-classically, we expect a non-zero vacuum expectation value of the scalar fields, $\langle\Phi\rangle\neq 0$ with $\langle\Phi\rangle^2=v^2$, or equivalently, $\langle\Phi^j\rangle=v\delta^{jN}$ modulo the $O(N)$ symmetry of the problem. Shifting the fields according to

$$\Phi^N(x) \;=\; v \;+\; \sigma(x),\quad \Phi^j(x) \;=\; \pi^j(x)\ (j<N), \tag{11}$$

and re-writing the Lagrangian in terms of the shifted fields, we obtain

$$\mathcal{L} \;=\; \tfrac{1}{2}(\partial\sigma)^2 \;-\; \mu^2\sigma^2 \;+\; \tfrac{1}{2}(\partial\underset{\sim}{\pi})^2 \;-\; \lambda v\sigma(\sigma^2+\underset{\sim}{\pi}^2) \;-\; \tfrac{1}{4}\lambda(\sigma^2+\underset{\sim}{\pi}^2)^2 + \text{ const} \tag{12}$$

where $\underset{\sim}{\pi}$ stands for the $(N-1)$–plet of the π^j fields, thus $\underset{\sim}{\pi}^2=\sum_j(\pi^j)^2$.

The free part of the Lagrangian (12) (the first 3 terms) describe one massive real scalar field $\sigma(x)$ of mass $m_\sigma = \sqrt{2}\mu$ and $(N-1)$ massless real scalars $\pi^j(x)$ which are the Goldstone particles of the $O(N)$ symmetry spontaneously broken down to $O(N-1)$ (thus $(N-1)$ broken symmetry generators, forming a vector multiplet of the unbroken $O(N-1)$ symmetry). Consequently, the non-zero contractions of the free σ and π fields are

$$\begin{aligned} \overbrace{\sigma(x)\,\sigma}(y) &= G^F(x-y)_{\text{mass}=m_\sigma}\,, \\ \overbrace{\pi^j(x)\,\pi^k}(y) &= \delta^{jk}\,G^F(x-y)_{\text{mass}=0}\,, \end{aligned} \tag{13}$$

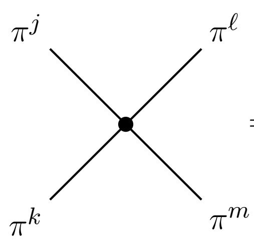
which give us two distinct Feynman propagators in the momentum basis,

$$\begin{aligned} \sigma \;=\!=\!=\!=\!=\!=\; \sigma \quad &= \frac{i}{q^2 - 2\mu^2 + i0}\,, \\ \pi^j \;\text{———————}\; \pi^k \quad &= \frac{i\delta^{jk}}{q^2 + i0}\,. \end{aligned} \tag{14}$$

The last two terms in the Lagrangian (12) give rise to the interaction Hamiltonian of the linear sigma model, namely

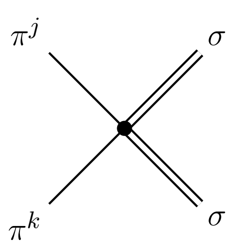
$$\hat{V} = \int d^3\mathbf{x} \Big(\lambda v \hat{\sigma}^3 + \lambda v \hat{\sigma} \hat{\underset{\sim}{\pi}}^2 + \tfrac{\lambda}{4} \hat{\sigma}^4 + \tfrac{\lambda}{2} \hat{\sigma}^2 \hat{\underset{\sim}{\pi}}^2 + \tfrac{\lambda}{4} (\hat{\underset{\sim}{\pi}}^2)^2 \Big). \tag{15}$$

The five terms in this interaction Hamiltonian give rise to five types of Feynman vertices. Proceeding exactly as in part (a) of the problem, we obtain

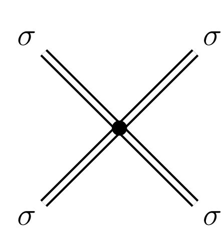


$$= -2i\lambda \big(\delta^{jk}\delta^{\ell m} + \delta^{j\ell}\delta^{km} + \delta^{jm}\delta^{k\ell} \big) \tag{16}$$

and similarly

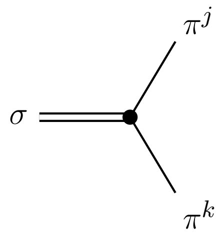


$$= -2i\lambda\delta^{ik} \qquad \text{and}$$

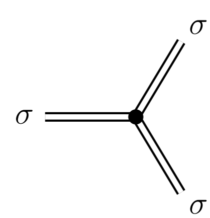


$$= -6i\lambda. \tag{17}$$

The remaining two vertices have valence = 3 and follow from the cubic terms in the interaction Hamiltonian (15). The analysis proceeds exactly as in the previous problem and yields



$= -2i\lambda v\delta^{jk}$ and

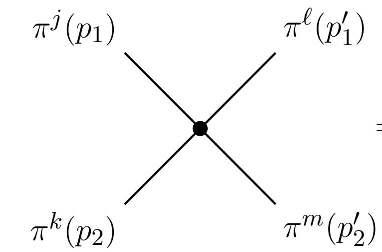


$= -6i\lambda v.$ (18)

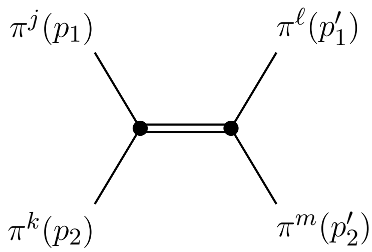
This completes the Feynman rules of the linear sigma model.

Textbook Problem **4.3**(c):

In this part of the problem, we use the Feynman rules we have just derived to calculate the tree-level $\pi\pi \to \pi\pi$ scattering amplitudes. As explained in class, a tree diagram ($L = 0$) with $E = 4$ external legs has either one valence $= 4$ vertex (and hence no propagators) or two valence $= 3$ vertices (and hence one propagator). Altogether, there are four such diagrams contributing to the tree-level $i\mathcal{M}\big(\pi^j(p_1) + \pi^k(p_2) \to \pi^\ell(p_1') + \pi^m(p_2')\big)$ — they are shown in the textbook. The diagrams evaluate to:

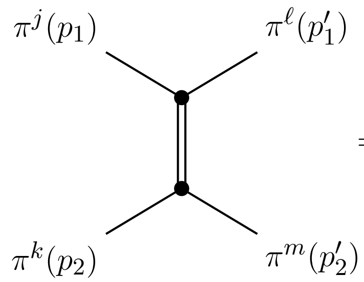


$$= \; -2i\lambda\big(\delta^{jk}\delta^{\ell m}+\delta^{j\ell}\delta^{km}+\delta^{jm}\delta^{k\ell}\big),$$

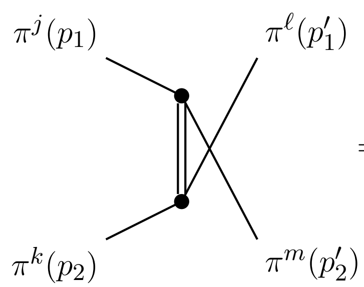


$$= \; (-2i\lambda v\delta^{jk})\,\frac{i}{(p_1+p_2)^2-2\mu^2}\,(-2i\lambda v\delta^{\ell m}),$$

(19)



$$= \; (-2i\lambda v\delta^{j\ell})\,\frac{i}{(p_1-p_1')^2-2\mu^2}\,(-2i\lambda v\delta^{km}),$$



$$= \; (-2i\lambda v\delta^{jm})\,\frac{i}{(p_1-p_2')^2-2\mu^2}\,(-2i\lambda v\delta^{k\ell}),$$

which gives the net scattering amplitude

$$\begin{aligned}
\mathcal{M}\big(\pi^j(p_1)+\pi^k(p_2)\to\pi^\ell(p_1')+\pi^m(p_2')\big) \;=\; & -2\lambda\delta^{jk}\delta^{\ell m}\left(1\,+\,\frac{2\lambda v^2}{(p_1+p_2)^2-2\mu^2}\right)\\
& -2\lambda\delta^{j\ell}\delta^{km}\left(1\,+\,\frac{2\lambda v^2}{(p_1-p_1')^2-2\mu^2}\right)\\
& -2\lambda\delta^{jm}\delta^{k\ell}\left(1\,+\,\frac{2\lambda v^2}{(p_1-p_2')^2-2\mu^2}\right).
\end{aligned} \tag{20}$$

Now, according to eq. (10) $\lambda v^2=\mu^2$, which makes for

$$\left(1\,+\,\frac{2\lambda v^2}{(p_1+p_2)^2-2\mu^2}\right) \;=\; \frac{(p_1+p_2)^2}{(p_1+p_2)^2-2\mu^2} \tag{21}$$

and ditto for the other two terms in the amplitude (20). Altogether, we now have

$$\mathcal{M} \;=\; -2\lambda\Big(\delta^{jk}\delta^{\ell m}\times\frac{(p_1+p_2)^2}{(p_1+p_2)^2-2\mu^2}\;+\;\delta^{j\ell}\delta^{km}\times\frac{(p_1-p'_1)^2}{(p_1-p'_1)^2-2\mu^2}\;+\;\delta^{jm}\delta^{k\ell}\times\frac{(p_1-p'_2)^2}{(p_1-p'_2)^2-2\mu^2}\Big), \tag{22}$$

which vanishes in the zero-momentum limit for *any one of the four pions*, initial or final. Indeed, since the pions are massless, $(p_1)^2=(p_2)^2=(p'_1)^2=(p'_2)^2=0$ and hence

$$\begin{aligned} s &\stackrel{\rm def}{=} (p_1+p_2)^2 \equiv (p'_1+p'_2)^2 = +2(p_1p_2) = +2(p'_1p'_2),\\ t &\stackrel{\rm def}{=} (p'_1-p_1)^2 \equiv (p'_2-p_2)^2 = -2(p'_1p_1) = -2(p'_2p_2),\\ u &\stackrel{\rm def}{=} (p'_1-p_2)^2 \equiv (p'_2-p'_1)^2 = -2(p_1p'_2) = -2(p'_1p_2), \end{aligned} \tag{23}$$

this whenever any one of the four momenta becomes small, all three numerators in the amplitude (22) become small as well, thus $\mathcal{M}=O(\text{small }p)$.

Please note that although eq. (22) gives only the tree-level approximation to the actual scattering amplitude, its behavior in the small pion momentum limit is correct and completely general. According to the *Goldstone theorem*, not only the Goldstone particles (such as 'pions' in this linear sigma model) are exactly massless, but also *any scattering amplitude involving any Goldstone particle vanishes as $O(p_\pi)$ when the Goldstone particle's momentum p_π goes to zero.*

To complete this part of the problem, let us now assume that all four pion's momenta are small compared to the σ-particle's mass $m_\sigma=\sqrt{2}\mu$. In this limit, all three denominators in eq. (22) are dominated by the $-2\mu^2$ term, hence

$$\mathcal{M} \;=\; \frac{1}{v^2}\left(\delta^{jk}\delta^{\ell m}(p_1+p_2)^2\;+\;\delta^{j\ell}\delta^{km}(p_1-p'_1)^2\;+\;\delta^{jm}\delta^{k\ell}(p_1-p'_2)^2\;+\;O\left(\frac{p^4}{m_\sigma^2}\right)\right). \tag{24}$$

For generic species of the four pions, this amplitude is of the order $O(p^2/v^2)$, but there is a cancellation when all for pions belong to the same species (this is unavoidable for $N=2$). Indeed,

$$\begin{aligned} (p_1+p_2)^2+(p_1-p'_1)^2+(p_1-p'_2)^2 &= 2(p_1p_2)-2(p_1p'_1)-2(p_1p'_2)\\ &= 2p_1(p_2-p'_1-p'_2=-p_1) \;=\; -2p_1^2=0 \end{aligned} \tag{25}$$

and hence

$$\mathcal{M}(\pi^1+\pi^1\to\pi^1+\pi^1) \;=\; \frac{1}{v^2}\left(0\;+\;O\left(\frac{p^4}{m_\sigma^2}\right)\right). \tag{26}$$

Finally, let us translate the amplitude (24) into the low-energy scattering cross sections:

$$\begin{aligned}
\frac{d\sigma(\pi^1+\pi^2\to\pi^1+\pi^2)}{d\Omega_{\text{c.m.}}} &= \frac{E^2_{\text{c.m.}}}{64\pi^2v^4}\times\sin^4\frac{\theta_{\text{c.m.}}}{2}\,,\\
\sigma_{\text{tot}}(\pi^1+\pi^2\to\pi^1+\pi^2) &= \frac{E^2_{\text{c.m.}}}{48\pi v^4}\,,\\
\frac{d\sigma(\pi^1+\pi^1\to\pi^2+\pi^2)}{d\Omega_{\text{c.m.}}} &= \frac{E^2_{\text{c.m.}}}{64\pi^2v^4}\,,\\
\sigma_{\text{tot}}(\pi^1+\pi^1\to\pi^2+\pi^2) &= \frac{E^2_{\text{c.m.}}}{32\pi v^4}\,,\\
\sigma(\pi^1+\pi^1\to\pi^1+\pi^1) &= O\left(\frac{E^6_{\text{c.m.}}}{v^4m^4_\sigma}\right).
\end{aligned}\tag{27}$$

<u>Textbook Problem **4.3**(d):</u>

The linear term $\Delta v = -a\Phi^{(N)}$ in the classical potential for the N scalar fields *explicitly* breaks the $O(N)$ symmetry of the theory. Hence the potential

$$V(\Phi) \;=\; \tfrac{1}{4}\lambda(\Phi^2)^2 \;-\; \tfrac{1}{2}\mu^2(\Phi^2) \;-\; a\Phi^{(N)} \tag{28}$$

now has a *non-degenerate minimum* at

$$\langle\Phi^j\rangle \;=\; v\delta^{jN} \qquad \text{where} \quad v \;\approx\; \sqrt{\frac{\mu}{\lambda}} \;+\; \frac{a}{2\mu} \;+O\left(\frac{a^2\sqrt{\lambda}}{\mu^2\sqrt{\mu}}\right). \tag{29}$$

Shifting the fields according to eq. (11) for the new value of v now gives us

$$\mathcal{L} \;=\; \tfrac{1}{2}(\partial\sigma)^2 \;-\; \tfrac{1}{2}m^2_\sigma\sigma^2 \;+\; \tfrac{1}{2}(\partial\underset{\sim}{\pi})^2 \;-\; \tfrac{1}{2}m^2_\pi\underset{\sim}{\pi}^2 -\; \lambda v\sigma(\sigma^2+\underset{\sim}{\pi}^2) \;-\; \tfrac{1}{4}\lambda(\sigma^2+\underset{\sim}{\pi}^2)^2 \tag{30}$$

(plus an irrelevant constant) where

$$m^2_\sigma \;=\; 2\mu^2 \;+\; 3m^2_\pi \qquad \text{and} \qquad m^2_\pi \;=\; \frac{a}{v} \;>\; 0. \tag{31}$$

Thus, the pions are no longer exactly massless Goldstone bosons but rather *pseudo–Goldstone* bosons with small but non-zero masses.

Comparing the Lagrangians (30) and (12) we immediately see identical interaction terms, hence the Feynman vertices of the modified sigma model are exactly as in eqs. (16), (17) and (18), without any modification (except for the new value of v). On the other hand, the Feynman propagators need adjustment to accommodate the new masses (31), thus

$$\begin{aligned} \sigma \;=\!=\!=\!=\!=\!=\; \sigma \;&=\; \frac{i}{q^2 - m_\sigma^2 + i0}\,, \\ \pi^j \;\text{———}\; \pi^k \;&=\; \frac{i\delta^{jk}}{q^2 - m_\pi^2 + i0}\,. \end{aligned} \tag{32}$$

The tree-level $\pi + \pi \to \pi + \pi$ scattering amplitude is governed by the same four Feynman diagrams as before, thus

$$\begin{aligned} \mathcal{M}\big(\pi^j(p_1) + \pi^k(p_2) \to \pi^\ell(p_1') + \pi^m(p_2')\big) \;=\; & -2\lambda\delta^{jk}\delta^{\ell m}\left(1 + \frac{2\lambda v^2}{(p_1+p_2)^2 - m_\sigma^2}\right) \\ & -2\lambda\delta^{j\ell}\delta^{km}\left(1 + \frac{2\lambda v^2}{(p_1-p_1')^2 - m_\sigma^2}\right) \\ & -2\lambda\delta^{jm}\delta^{k\ell}\left(1 + \frac{2\lambda v^2}{(p_1-p_2')^2 - m_\sigma^2}\right), \end{aligned} \tag{33}$$

exactly as in eq. (20), except for the new v and new m_σ^2. The exact equation for the minimum (29) is

$$\lambda v^2 \;-\; \mu^2 \;=\; \frac{a}{v} \;=\; m_\pi^2 \tag{34}$$

hence

$$2\lambda v^2 \;-\; m_\sigma^2 \;=\; -m_\pi^2 \tag{35}$$

and

$$\left(1 + \frac{2\lambda v^2}{(p_1+p_2)^2 - m_\sigma^2}\right) \;=\; \frac{(p_1+p_2)^2 - m_\pi^2}{(p_1+p_2)^2 - m_\sigma^2} \tag{36}$$

and ditto for the other two terms in the amplitude (33). Therefore, instead of eq. (22) we now have

$$\begin{aligned} \mathcal{M} \;=\; -2\lambda\bigg(& \delta^{jk}\delta^{\ell m} \times \frac{(p_1+p_2)^2 - m_\pi^2}{(p_1+p_2)^2 - m_\sigma^2} \;+\; \delta^{j\ell}\delta^{km} \times \frac{(p_1-p_1')^2 - m_\pi^2}{(p_1-p_1')^2 - m_\sigma^2} \\ & +\; \delta^{jm}\delta^{k\ell} \times \frac{(p_1-p_2')^2 - m_\pi^2}{(p_1-p_2')^2 - m_\sigma^2}\bigg), \end{aligned} \tag{37}$$

which in the low-energy limit $E_{\text{c.m.}} \ll m_\sigma$ simplifies to

$$\begin{aligned} \mathcal{M} \;=\; \left(\frac{2\lambda}{m_\sigma^2} \approx \frac{1}{v^2}\right)\bigg(& \delta^{jk}\delta^{\ell m}\left((p_1+p_2)^2 - m_\pi^2\right) \;+\; \delta^{j\ell}\delta^{km}\left((p_1-p_1')^2 - m_\pi^2\right) \\ & +\; \delta^{jm}\delta^{k\ell}\left((p_1-p_2')^2 - m_\pi^2\right) \;+\; O\Big(\frac{p^4}{m_\sigma^2}\Big)\bigg). \end{aligned} \tag{38}$$

In particular, near the threshold $(p_1+p_2)^2 = E_{\text{c.m.}}^2 \approx 4m_\pi^2$ while $(p_1'-p_1)^2 \approx (p_2'-p_1)^2 \approx 0$

and hence

$$\mathcal{M} \;\approx\; \frac{m_\pi^2}{v^2}\times\Big(3\delta^{jk}\delta^{\ell m}\;-\;\delta^{j\ell}\delta^{km}\;-\;\delta^{jm}\delta^{k\ell}\Big). \tag{39}$$

This threshold amplitude does not vanish. Instead,

$$\mathcal{M}\;\sim\;\frac{m_\pi^2}{v^2}\;=\;\frac{a}{v^3}\,. \tag{40}$$

<u>Problem **2**</u>:

In Pauli-Villars (PV) regularization the scalar propagators in the loop integral are UV-softened as discussed in class. The product of the two propagators in the loop diagram becomes

$$\frac{1}{q_1^2+m^2}\times\frac{1}{q_2^2+m^2}\;-\;\frac{1}{q_1^2+\Lambda^2}\times\frac{1}{q_2^2+m^2}\;-\;\frac{1}{q_1^2+m^2}\times\frac{1}{q_2^2+\Lambda^2}\;+\;\frac{1}{q_1^2+\Lambda^2}\times\frac{1}{q_2^2+\Lambda^2} \tag{41}$$

where $q_2 = k - q_1$. Applying Feynman's parameter trick to each of these products, we obtain

$$\int\limits_0^1\!dx\left(\frac{1}{[q^2+\Delta]^2}\;-\;\frac{1}{[q^2+\Delta+x\tilde{\Lambda}^2]^2}\;-\;\frac{1}{[q^2+\Delta+(1-x)\tilde{\Lambda}^2]^2}\;+\;\frac{1}{[q^2+\Delta+\tilde{\Lambda}^2]^2}\right) \tag{42}$$

where $q = q_1 - kx$ is the same in all four terms,

$$\Delta(x)\;=\;m^2\;+\;x(1-x)k_E^2\;=\;m^2\;-\;x(1-x)k_{\rm Mink}^2 \tag{43}$$

is also the same in all the terms, and finally

$$\tilde{\Lambda}^2\;=\;\Lambda^2\;-\;m^2\;\approx\;\Lambda^2 \tag{44}$$

is what makes the four terms different from each other.

Now we need to integrate the propagator product over the Euclidean momentum. As in class, we integrate over the momentum before integrating over x in eq. (42), and this allows us to shift the integration variable from q_1 (or q_2) to q and use spherical symmetry. Thus,

$$d^4q_E\;=\;2\pi^2\,q^3\,dq\;=\;\pi^2\,q^2\,dq^2, \tag{45}$$

and therefore

$$\int\frac{d^4q_E}{(2\pi)^4}\left(\frac{1}{[q^2+\Delta]^2}\;-\;\frac{1}{[q^2+\Delta+x\tilde\Lambda^2]^2}\;-\;\frac{1}{[q^2+\Delta+(1-x)\tilde\Lambda^2]^2}\;+\;\frac{1}{[q^2+\Delta+\tilde\Lambda^2]^2}\right)$$

$$= \frac{1}{16\pi^2}\int\limits_0^\infty dq^2\left[\begin{array}{ll}\dfrac{q^2}{[q^2+\Delta]^2} & -\;\dfrac{q^2}{[q^2+\Delta+x\tilde\Lambda^2]^2}\\ & -\;\dfrac{q^2}{[q^2+\Delta+(1-x)\tilde\Lambda^2]^2}\\ & +\;\dfrac{q^2}{[q^2+\Delta+\tilde\Lambda^2]^2}\end{array}\right]$$

$$= \frac{1}{16\pi^2}\left(\begin{array}{l}\left(\log(q^2+\Delta)\;-\;\dfrac{q^2}{q^2+\Delta}\right)\\ -\;\left(\log(q^2+\Delta+x\tilde\Lambda^2)\;-\;\dfrac{q^2}{q^2+\Delta+x\tilde\Lambda^2}\right)\\ -\;\left(\log(q^2+\Delta+(1-x)\tilde\Lambda^2)\;-\;\dfrac{q^2}{q^2+\Delta+(1-x)\tilde\Lambda^2}\right)\\ +\;\left(\log(q^2+\Delta+\tilde\Lambda^2)\;-\;\dfrac{q^2}{q^2+\Delta+\tilde\Lambda^2}\right)\end{array}\right)\Bigg|_{q^2=0}^{q^2=\infty}$$

$$= \log\frac{(\Delta+x\tilde\lambda^2)\times(\Delta+(1-x)\tilde\Lambda^2)}{\Delta\times(\Delta+\tilde\Lambda^2)}$$

$$\approx \log\frac{x\tilde\Lambda^2\times(1-x)\tilde\lambda^2}{\Delta\times\tilde\Lambda^2}$$

$$\approx \log\frac{x(1-x)\Lambda^2}{\Delta}\,. \tag{46}$$

Consequently, the whole diagram evaluates to

$$\begin{aligned}\mathcal{M} &= \frac{\lambda^2}{2}\times\frac{1}{16\pi^2}\int\limits_0^1 dx\,\log\frac{x(1-x)\Lambda^2}{\Delta=m^2-x(1-x)k^2}\\ &= \frac{\lambda^2}{32\pi^2}\left[\log\frac{\Lambda^2}{m^2}\;+\int\limits_0^1 dx\,\log x(1-x)\;-\int\limits_0^1 dx\,\log\left(1\;-\;x(1-x)\frac{k^2}{m^2}\right)\right]\\ &= \frac{\lambda^2}{32\pi^2}\left[\log\frac{\Lambda^2}{m^2}\;-\;2\;+\;I(k^2/m^2)\right].\end{aligned} \tag{47}$$

Finally, let us compare our result (47) for the PV regulator with

$$\mathcal{M} \;=\; \frac{\lambda^2}{32\pi^2}\left[\log\frac{\Lambda^2}{m^2}\;-\;1\;+\;I(k^2/m^2)\right] \tag{48}$$

which one can similarly obtain a sharp (S) UV cutoff. Clearly the only difference between the two formulæ is the numerical constant inside the square brackets. Moreover, this difference may be absorbed into a re-definition of the UV cutoff parameter: If we set

$$\Lambda^2_{\rm PV} \;=\; \exp(1)\times\Lambda^2_{\rm S}\,, \tag{49}$$

then

$$\log\frac{\Lambda^2_{PV}}{m^2}\;-\;2\;+\;I(k^2/m^2)\;=\;\log\frac{\Lambda^2_S}{m^2}\;-\;1\;+\;I(k^2/m^2) \tag{50}$$

and eqs. (47) and (48) are in perfect agreement.

Problem **3**(a):

We begin with the muon decay amplitude

$$\mathcal{M}(\mu^-\to e^-\nu_\mu\bar\nu_e)\;=\;\frac{G_F}{\sqrt{2}}\left[\bar u(\nu_\mu)(1-\gamma^5)\gamma^\alpha u(\mu^-)\right]\times\left[\bar u(e^-)(1-\gamma^5)\gamma_\alpha v(\bar\nu_e)\right], \tag{3}$$

Its complex conjugate can be written as

$$\mathcal{M}^*\;=\;\frac{G_F}{\sqrt{2}}\left[\bar u(\mu^-)\gamma^\beta(1+\gamma^5)u(\nu_\mu)\right]\times\left[\bar v(\bar\nu_e)\gamma_\beta(1+\gamma^5)u(e^-)\right], \tag{51}$$

where $(1-\gamma^5)$ factors become $(1+\gamma^5)$ because $\bar\gamma^5\equiv\gamma^0(\gamma^5)^\dagger\gamma^0=-\gamma^5$. Consequently,

$$\begin{aligned}|\mathcal{M}|^2\;=\;\tfrac{1}{2}G_F^2\,&\left[\bar u(\nu_\mu)(1-\gamma^5)\gamma^\alpha u(\mu^-)\bar u(\mu^-)\gamma^\beta(1+\gamma^5)u(\nu_\mu)\right]\\ &\times\left[\bar u(e^-)(1-\gamma^5)\gamma_\alpha v(\bar\nu_e)\bar v(\bar\nu_e)\gamma_\beta(1+\gamma^5)u(e^-)\right]\end{aligned} \tag{52}$$

and hence

$$\begin{aligned}\tfrac{1}{2}\sum_{\substack{\rm all\\ \rm spins}}|\mathcal{M}|^2\;=\;\tfrac{1}{4}G_F^2\,&{\rm tr}\Big((1-\gamma^5)\gamma^\alpha(\not p_\mu+M_\mu)\gamma^\beta(1+\gamma^5)(\not p_\nu+m_\nu)\Big)\\ &\times{\rm tr}\Big((1-\gamma^5)\gamma_\alpha(\not p_{\bar\nu}-m_{\bar\nu})\gamma_\beta(1+\gamma^5)(\not p_e+m_e)\Big).\end{aligned} \tag{53}$$

Please note that here and henceforth the indices μ, e $\nu\equiv\nu_\mu$, and $\bar\nu\equiv\bar\nu_e$ denote the particles to which respective momenta belong and have nothing to do with the Lorentz indices of those momenta. For the Lorentz indices, I use here α, β and later also γ, δ, σ and ρ. Thus, $p_{\mu\alpha}$ is the α's component of the muon's 4–momentum, *etc.*, *etc.*

Having derived eq. (53), we now need to evaluate the traces. For the first trace, we eliminate terms containing odd numbers of γ^ρ matrices and write

$$\begin{aligned}
\operatorname{tr}\Big((1-\gamma^5)\gamma^\alpha(\not{p}_\mu+M_\mu)\gamma^\beta(1+\gamma^5)(\not{p}_\nu+m_\nu)\Big) &= \\
&= \operatorname{tr}\Big((1-\gamma^5)\gamma^\alpha\,\not{p}_\mu\gamma^\beta(1+\gamma^5)\,\not{p}_\nu\Big) \;+\; \operatorname{tr}\Big((1-\gamma^5)\gamma^\alpha M_\mu\gamma^\beta(1+\gamma^5)m_\nu\Big) \\
&= \operatorname{tr}\Big((1-\gamma^5)\gamma^\alpha\,\not{p}_\mu\gamma^\beta\,\not{p}_\nu(1-\gamma^5)\Big) \;+\; M_\mu m_\nu\,\operatorname{tr}\Big((1-\gamma^5)\gamma^\alpha\gamma^\beta(1+\gamma^5)\Big) \\
&= \operatorname{tr}\Big((1-\gamma^5)^2\gamma^\alpha\,\not{p}_\mu\gamma^\beta\,\not{p}_\nu\Big) \;+\; M_\mu m_e\,\operatorname{tr}\Big((1+\gamma^5)(1-\gamma^5)\gamma^\alpha\gamma^\beta\Big) \\
&= 2\operatorname{tr}\Big((1-\gamma^5)\gamma^\alpha\,\not{p}_\mu\gamma^\beta\,\not{p}_\nu\Big) \;+\; 0 \\
&= 2\operatorname{tr}\Big(\gamma^\alpha\,\not{p}_\mu\gamma^\beta\,\not{p}_\nu\Big) \;-\; 2\operatorname{tr}\Big(\gamma^5\gamma^\alpha\,\not{p}_\mu\gamma^\beta\,\not{p}_\nu\Big) \\
&= 8\Big[p_\mu^\alpha p_\nu^\beta \,+\, p_\mu^\beta p_\nu^\alpha \,-\, g^{\alpha\beta}(p_\mu\cdot p_\nu)\Big] \;+\; 8i\epsilon^{\alpha\gamma\beta\delta}p_{\mu\gamma}p_{\nu\delta}. \qquad (54)
\end{aligned}$$

Similarly, the second trace evaluates to

$$\begin{aligned}
\operatorname{tr}\big((1-\gamma^5)\gamma_\alpha(\not{p}_e+m_e)\gamma_\beta(1+\gamma^5)(\not{p}_{\bar\nu}-m_{\bar\nu}\big) &= \qquad (55) \\
&= 8\big[(p_{e\alpha}p_{\bar\nu\beta} \,+\, p_{e\beta}p_{\bar\nu\alpha} \,-\, g_{\alpha\beta}(p_e\cdot p_{\bar\nu})\big] \;+\; 8i\epsilon_{\alpha\rho\beta\sigma}p_{\bar\nu}^\rho p_e^\sigma\,.
\end{aligned}$$

It remains to substitute the trace formulæ (54) and (55) back into eq. (53) and contract the Lorentz indices. Thus,

$$\begin{aligned}
\tfrac{1}{2}\sum_{\substack{\text{all}\\\text{spins}}}|\mathcal{M}|^2 \;=\; 16G_F^2 &\Big(\Big[p_\mu^\alpha p_\nu^\beta \,+\, p_\mu^\beta p_\nu^\alpha \,-\, g^{\alpha\beta}(p_\mu\cdot p_\nu)\Big] + i\epsilon^{\alpha\gamma\beta\delta}p_{\mu\gamma}p_{\nu\delta}\Big) \\
\times &\Big(\Big[p_{e\alpha}p_{\bar\nu\beta} \,+\, p_{e\beta}p_{\bar\nu\alpha} \,-\, g_{\alpha\beta}(p_e\cdot p_{\bar\nu})\Big] + i\epsilon_{\alpha\rho\beta\sigma}p_{\bar\nu}^\rho p_e^\sigma\Big)
\end{aligned}$$

⟪using symmetry/antisymmetry of factors under $\alpha\leftrightarrow\beta$⟫

$$\begin{aligned}
&= 16G_F^2\Big(\Big[p_\mu^\alpha p_\nu^\beta \,+\, p_\mu^\beta p_\nu^\alpha \,-\, g^{\alpha\beta}(p_\mu\cdot p_\nu)\Big] \times \Big[p_{e\alpha}p_{\bar\nu\beta} \,+\, p_{e\beta}p_{\bar\nu\alpha} \,-\, g_{\alpha\beta}(p_e\cdot p_{\bar\nu})\Big] \\
&\qquad -\; \epsilon^{\alpha\gamma\beta\delta}p_{\mu\gamma}p_{\nu\delta}\times\epsilon_{\alpha\rho\beta\sigma}p_{\bar\nu}^\rho p_e^\sigma\Big) \\
&= 16G_F^2\Big(\Big[2(p_\mu\cdot p_e)(p_\nu\cdot p_{\bar\nu}) \,+\, 2(p_\mu\cdot p_{\bar\nu})(p_\nu\cdot p_e) \\
&\qquad -\, 2(p_\mu\cdot p_\nu)(p_e\cdot p_{\bar\nu}) \,-\, 2(p_\mu\cdot p_\nu)(p_e\cdot p_{\bar\nu}) \,+\, 4(p_\mu\cdot p_\nu)(p_e\cdot p_{\bar\nu})\Big] \\
&\qquad +\; \Big[2(p_\mu\cdot p_{\bar\nu})(p_\nu\cdot p_e) \,-\, 2(p_\mu\cdot p_e)(p_\nu\cdot p_{\bar\nu})\Big]\Big) \\
&= 64G_F^2\,(p_\mu\cdot p_{\bar\nu})(p_\nu\cdot p_e)\,. \qquad (56)
\end{aligned}$$

Problem **3**(b):

As explained in the *Peskin & Schroeder* textbook, the partial rate of a decay process (in the rest frame of the initial particle) is given by

$$d\Gamma \;=\; \frac{1}{2M_0} \times \overline{|\mathcal{M}|^2} \times d\mathcal{P} \tag{57}$$

where $\mathcal{M}$ is the decay's amplitude, $\overline{|\mathcal{M}|^2}$ is $|\mathcal{M}|^2$ averaged over the unknown initial spins and summed over the unmeasured final spins, and $d\mathcal{P}$ is the infinitesimal phase space factor for the final particles. For three final particles,

$$d\mathcal{P} \;=\; \frac{d^3\mathbf{p}_1}{(2\pi)^3(2E_1)}\,\frac{d^3\mathbf{p}_2}{(2\pi)^3(2E_2)}\,\frac{d^3\mathbf{p}_3}{(2\pi)^3(2E_3)} \times (2\pi)^3\delta^{(3)}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3) \times (2\pi)\delta(E_1+E_2+E_3-M_0) \tag{58}$$

where the energy-momentum conservation law apply in the rest frame, thus $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = \mathbf{p}_{\rm tot} = \mathbf{0}$ and $E_1 + E_2 + E_3 = E_{\rm tot} = M_0$.

We start by using the momentum-conservation δ–function to eliminate eliminate the $\mathbf{p}_3$ as independent variable, thus

$$d\mathcal{P} \;=\; \frac{d^3\mathbf{p}_1\, d^3\mathbf{p}_2}{256\pi^5}\; \frac{\delta(E_1 + E_2 + E_3 - E_{\rm tot})}{E_1 E_2 E_3}\bigg|_{\mathbf{p}_3=-(\mathbf{p}_1+\mathbf{p}_2)}\,. \tag{59}$$

Next, we use spherical coordinates for the two remaining momenta,

$$d^3\mathbf{p}_1 \;=\; p_1^2\, dp_1\, d^2\Omega_1\,, \qquad d^3\mathbf{p}_2 \;=\; p_2^2\, dp_2\, d^2\Omega_2\,, \tag{60}$$

and then replace the $d^2\Omega_2$ describing the direction of the second particle's momentum relative to the fixed external frame with

$$d^2\Omega_2^{(1)} \;=\; d\theta_{12}\, \sin\theta_{12}\, d\phi_2^{(1)}$$

describing the same direction of $\mathbf{p}_2$ relative to the frame centered on the $\mathbf{p}_1$. Consequently,

$$d^2\Omega_1\, d^2\Omega_2 \;=\; d^2\Omega_1\, d^2\Omega_2^{(1)} \;=\; \left[d^2\Omega_1\, d\phi_2^{(1)}\right] d\theta_{12}\, \sin\theta_{12} \;\equiv\; d^3\Omega \times d(\cos\theta_{12}) \tag{61}$$

and hence

$$d\mathcal{P} \;=\; \frac{d^3\Omega}{256\pi^5} \times \frac{p_1^2 p_2^2}{E_1 E_2 E_3}\, dp_1\, dp_2\, d(\cos\theta_{12})\, \delta(E_1 + E_2 + E_3 - E_{\rm tot})\bigg|_{\mathbf{p}_3=-(\mathbf{p}_1+\mathbf{p}_2)}\,. \tag{62}$$

Next, we use the cosine theorem

$$p_3^2 \;=\; (\mathbf{p}_1 + \mathbf{p}_2)^2 \;=\; p_1^2 \,+\, p_2^2 \,+\, 2p_1p_2\cos\theta_{12}$$

which gives

$$d(\cos\theta_{12}) \;=\; \frac{p_3\,dp_3}{p_1\,p_2}$$

(for fixed p_1, p_2) and therefore

$$d\mathcal{P} \;=\; \frac{d^3\Omega}{256\pi^5} \times \frac{p_1p_2p_3}{E_1E_2E_3} \times dp_1\,dp_2\,dp_3 \times \delta(E_1+E_2+E_3-E_{\text{tot}}). \tag{63}$$

Finally, we notice that for a relativistic particle of any mass $pdp = EdE$, hence

$$d\mathcal{P} \;=\; \frac{d^3\Omega}{256\pi^5} \times dE_1\,dE_2\,dE_3\,\delta(E_1+E_2+E_3-E_{\text{tot}}), \tag{64}$$

and therefore eq. (57) for the partial decay rate.

<u>Problem **3**(c)</u>:

It remains to determine the limits of kinematically allowed ways to distribute the net energy $E_{\text{tot}} = M_0$ of the process among the three final particles. Such limits follow from the triangle inequalities for the three momenta,

$$p_1 \;\le\; p_2 \,+\, p_3\,, \qquad p_2 \;\le\; p_1 \,+\, p_3\,, \qquad p_3 \;\le\; p_1 \,+\, p_1\,, \tag{65}$$

which look simple but produce rather complicated inequalities for the energies $E_1 = \sqrt{p_1^2+m_1^2}$, $E_2 = \sqrt{p_2^2+m_2^2}$, and $E_3 = \sqrt{p_3^2+m_1^3}$. However, when all three final particles are massless, the kinematic restrictions become simply

$$E_1 \;\le E_2 \,+\, E_3 \;=\; M_0 \;-\; E_1 \tag{66}$$

and ditto for the other two inequalities, or equivalently

$$0 \;\le\; E_1, E_2, E_3 \;\le\; \tfrac{1}{2}M_0\,, \qquad \text{while} \quad E_1+E_2+E_3 \;=\; M_0\,. \tag{5}$$

Problem $\mathbf{3}$(d):

In light of eqs. (3) and (57), the partial decay rate of the muon at rest is given by

$$d\Gamma(\mu^- \to e^-\nu_\mu\bar{\nu}_e) \;=\; \frac{G_F^2}{8\pi^5 M_\mu}\,(p_\mu\cdot p_{\bar\nu})(p_e\cdot p_\nu)\times dE_e\,dE_\nu\,dE_{\bar\nu}\,d^3\Omega\,\delta(E_e+E_\nu+E_{\bar\nu}-M_\mu). \tag{67}$$

Specializing to the muon's frame, we have

$$(p_\mu\cdot p_{\bar\nu}) \;=\; M_\mu E_{\bar\nu} \tag{68}$$

while

$$\begin{aligned}(p_e\cdot p_e) \;&=\; E_eE_\nu \;-\; p_ep_\nu\cos\theta_{e\nu}\\ &=\; E_eE_\nu \;+\; \tfrac12 p_e^2 \;+\; \tfrac12 p_\nu^2 \;-\; \tfrac12 p_{\bar\nu}^2\\ &\langle\!\langle\text{neglecting } m_e, m_\nu, m_{\bar\nu}\rangle\!\rangle\\ &=\; E_eE_\nu \;+\; \tfrac12 E_e^2 \;+\; \tfrac12 E_\nu^2 \;-\; \tfrac12 E_{\bar\nu}^2\\ &=\; \tfrac12(E_e+E_\nu)^2 \;-\; \tfrac12 E_{\bar\nu}^2\ \langle\!\langle\text{using } E_e+E_\nu = M_\mu - E_{\bar\nu}\rangle\!\rangle\\ &=\; \tfrac12 M_\mu(M_\mu-2E_{\bar\nu}),\end{aligned} \tag{69}$$

Hence,

$$d\Gamma(\mu^- \to e^-\nu_\mu\bar{\nu}_e) \;=\; \frac{G_F^2}{16\pi^5}\,M_\mu E_{\bar\nu}(M_\mu-2E_{\bar\nu})\times dE_e\,dE_\nu\,dE_{\bar\nu}\,d^3\Omega\,\delta(E_e+E_\nu+E_{\bar\nu}-M_\mu). \tag{70}$$

At this point we are ready to integrate over the final-state variables. In light of $\int d^3\Omega = 8\pi^2$ and the kinematic limits (5), we immediately obtain

$$\begin{aligned}\Gamma(\mu^- \to e^-\nu_\mu\bar{\nu}_e) \;&=\; \frac{G_F^2 M_\mu}{2\pi^3}\iiint_0^{\frac12 M_\mu} dE_e\,dE_{\bar\nu}\,dE_\nu\,E_{\bar\nu}(M_\mu-2E_{\bar\nu})\delta(E_e+E_\nu+E_{\bar\nu}-M_\mu)\\ &=\; \frac{G_F^2 M_\mu}{2\pi^3}\int_0^{\frac12 M_\mu} dE_e \int_{\frac12 M_\mu - E_e}^{\frac12 M_\mu} dE_{\bar\nu}\,E_{\bar\nu}(M_\mu-2E_{\bar\nu})\\ &=\; \frac{G_F^2 M_\mu}{2\pi^3}\int_0^{\frac12 M_\mu} dE_e\,E_e^2(\tfrac12 M_\mu - \tfrac23 E_e).\end{aligned} \tag{71}$$

In other words, the partial muon decay rate with respect to the final electron's energy is given by

$$\frac{d\Gamma}{dE_e} \;=\; \frac{G_F^2 M_\mu}{12\pi^3} \times E_e^2 (3M_\mu - 4E_e) \tag{72}$$

or rather

$$\frac{d\Gamma}{dE_e} \;\approx\; \begin{cases} \frac{G_F^2}{12\pi^3}\, M_\mu E_e^2 (3M_\mu - 4E_e) & \text{for } E_e < \frac{1}{2} M_\mu, \\ 0 & \text{for } E_e > \frac{1}{2} M_\mu. \end{cases} \tag{73}$$

It remains to calculate the total decay rate of the muon by integrating the partial rate (73) over the electron's energy. The result is

$$\Gamma_{\text{tot}}(\mu \to e\nu\bar{\nu}) \;=\; \frac{G_F^2 M_\mu^5}{192\pi^3}\,. \tag{74}$$