COMPUTATION OF TWO-BODY MATRIX ELEMENTS FROM WIRINGA'S V-18 POTENTIAL.

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In this document we discuss the computation of two-body matrix elements from the "V-18" interaction published by Wiringa[1]. We first discuss the symmetries of matrix elements.

Two-body matrix elements can be specified in particle-particle coupling or in particle-hole coupling. Both of the matrix elements completely specify the interaction, and either set can be calculated from the other set. We first discuss the symmetries and the angular momentum couplings:

1. Symmetries of two-body matrix elements

Generally we use the definition: $k = j + 1/2$ and $\hat{j} = \sqrt{2j+1}$. The matrix element in m-representation can be written as

$$\langle m_a\ m_b\ |V|\ m_c\ m_d\rangle\ = V_{m_a m_b, m_c m_d} = \langle m_a m_c^{-1}|V|m_d m_b^{-1}\rangle = \langle m_a \bar{m}_c|V|m_d \bar{m}_b\rangle \qquad 1.1$$

We use these notations interchangeably. The angular momentum coupled matrix element is constructed as

$$\langle (j_a j_b)_L\ |V^L|\ (j_c j_d)_L\rangle = \sum_{m_a m_b m_c m_d} \langle j_a m_a j_b m_b|LM\rangle\langle j_c m_c j_d m_d|LM\rangle\langle m_a m_b|V|m_c m_d\rangle \qquad 1.2$$

which is independent of M. It has the symmetry:

$$\langle (j_a j_b)_L\ |V^L|\ (j_c j_d)_L\rangle = (-)^{k_a+k_b+k_c+k_d}\langle (j_b j_a)_L\ |V^L|\ (j_d j_c)_L\rangle$$

The inverse relation is

$$\langle m_a m_b|V|m_c m_d\rangle = \sum_{L,M} \langle j_a m_a j_b m_b|LM\rangle\langle j_c m_c j_d m_d|LM\rangle\langle (j_a j_b)_L|V^L|(j_c j_d)_L\rangle \qquad 1.3$$

Instead of using this particle-particle(pp) coupling one can use particle-hole(ph) coupling to construct angular momentum coupled matrix elements. The (ph)-coupled matrix element is defined as

$$\langle (j_1\bar{j}_2)_\lambda|V^\lambda|(j_3\bar{j}_4)_\lambda\rangle = \sum_{m_1 m_2 m_3 m_4} (-)^{k_2+m_2+k_4-m_4}\langle j_1 m_1 j_2 - m_2|\lambda\mu\rangle\langle j_3 m_3 j_4 - m_4|\lambda\mu\rangle\langle m_1 m_2^{-1}|V|m_3 m_4^{-1}\rangle \qquad 1.4$$

These are again independent of μ. The matrix elements in pp-coupling are related to the matrix elements in ph-coupling through[2]

$$\langle (j_1\bar{j}_2)_\lambda|V^\lambda|(j_3\bar{j}_4)_\lambda\rangle = \sum_L (-)^{k_2+k_3+L+1}(2L+1)\begin{Bmatrix} j_1 & j_2 & \lambda \\ j_3 & j_4 & L \end{Bmatrix}\langle (j_1 j_4)_L|V^L|(j_2 j_3)_L\rangle \qquad 1.5$$

This is obtained via general recoupling, using:

$$\begin{aligned}
&\sum_M \langle j_1 m_1 j_2 m_2|LM\rangle\langle j_3 m_3 j_4 m_4|LM\rangle = \\
&=(-)^{j_2+m_1+j_4+m_3}\sum_{\lambda,\mu}(2L+1)\begin{Bmatrix} j_1 & j_2 & L \\ j_3 & j_4 & \lambda \end{Bmatrix}\langle j_1 m_1 j_4 - m_4|\lambda\mu\rangle\langle j_3 m_3 j_2 - m_2|\lambda\mu\rangle \\
&=(-)^{j_2-m_2-j_4-m_3-L}\sum_{\lambda,\mu}(2L+1)\begin{Bmatrix} j_1 & j_2 & L \\ j_4 & j_3 & \lambda \end{Bmatrix}\langle j_1 m_1 j_3 - m_3|\lambda\mu\rangle\langle j_4 m_4 j_2 - m_2|\lambda\mu\rangle
\end{aligned}$$

The inverse relation for the matrix elements is

$$\langle (j_1 j_4)_L | V^L | (j_2 j_3)_L \rangle = (-)^{L+1} \sum_{\lambda} (-)^{k_2+k_3} (2\lambda+1) \begin{Bmatrix} j_1 & j_2 & \lambda \\ j_3 & j_4 & L \end{Bmatrix} \langle (j_1 \bar{j}_2)_\lambda | V^\lambda | (j_3 \bar{j}_4)_\lambda \rangle \qquad 1.6$$

Anti-symmetric matrix elements are formed in m-representation as

$$\langle m_a m_b | V^a | m_c m_d \rangle = \langle m_a m_b | V | m_c m_d \rangle - \langle m_a m_b | V | m_d m_c \rangle \qquad 1.7$$

In (pp) angular momentum coupling this becomes

$$\langle (j_a j_b)_L | V^{aL} | (j_c j_d)_L \rangle = \langle (j_a j_b)_L | V^L | (j_c j_d)_L \rangle + (-)^{L+k_c+k_d} \langle (j_a j_b)_L | V^L | (j_d j_c)_L \rangle \qquad 1.8$$

These anti-symmetric matrix elements have the symmetry

$$\langle (j_a j_b)_L | V^{aL} | (j_c j_d)_L \rangle = (-)^{L+k_c+k_d} \langle (j_a j_b)_L | V^{aL} | (j_d j_c)_L \rangle = (-)^{L+k_a+k_b} \langle (j_b j_a)_L | V^{aL} | (j_c j_d)_L \rangle \qquad 1.9$$

In (ph)-coupling the anti-symmetrization can be obtained by transforming the anti-symmetric matrix elements in (pp)-coupling using (1.5). Thus we obtain

$$\begin{aligned} \langle (j_1 \bar{j}_2)_\lambda | V^{a\lambda} | (j_3 \bar{j}_4)_\lambda \rangle = {} & \langle (j_1 \bar{j}_2)_\lambda | V^\lambda | (j_3 \bar{j}_4)_\lambda \rangle + \\ & \sum_{\lambda'} (-)^{\lambda+\lambda'} (-)^{k_2+k_3} (2\lambda'+1) \begin{Bmatrix} j_1 & j_2 & \lambda \\ j_4 & j_3 & \lambda' \end{Bmatrix} \langle (j_1 \bar{j}_3)_{\lambda'} | V^{\lambda'} | (j_2 \bar{j}_4)_{\lambda'} \rangle \end{aligned} \qquad 1.10$$

The symmetry of the relations can be significantly improved by introducing the "Ring"-phase. For (pp)-coupling the phase is

$$\langle (j_a j_b)_L | V^{aLR} | (j_c j_d)_L \rangle = (-)^{k_a+k_c} \langle (j_a j_b)_L | V^{aL} | (j_c j_d)_L \rangle \qquad 1.11$$

with this extra phase the matrix elements have the symmetry

$$\langle (j_a j_b)_L | V^{aLR} | (j_c j_d)_L \rangle = (-)^L \langle (j_a j_b)_L | V^{aLR} | (j_d j_c)_L \rangle = (-)^L \langle (j_b j_a)_L | V^{aLR} | (j_c j_d)_L \rangle \qquad 1.12$$

and the anti-symmetric matrix elements can be calculated as

$$\langle (j_a j_b)_L | V^{aLR} | (j_c j_d)_L \rangle = \langle (j_a j_b)_L | V^{LR} | (j_c j_d)_L \rangle + (-)^L \langle (j_a j_b)_L | V^{LR} | (j_d j_c)_L \rangle \qquad 1.13$$

For matrix elements in (ph)-coupling the "Ring"-phase is[3]

$$\langle (j_1 \bar{j}_2)_\lambda | V^{a\lambda R} | (j_3 \bar{j}_4)_\lambda \rangle = (-)^{k_1+k_3} \langle (j_1 \bar{j}_2)_\lambda | V^{a\lambda} | (j_3 \bar{j}_4)_\lambda \rangle \qquad 1.14$$

With this phase the anti-symmetric matrix elements have the symmetry

$$\begin{aligned} \langle (j_1 \bar{j}_2)_\lambda | V^{a\lambda R} | (j_3 \bar{j}_4)_\lambda \rangle &= \langle (j_2 \bar{j}_1)_\lambda | V^{a\lambda R} | (j_4 \bar{j}_3)_\lambda \rangle \\ &= \sum_{\lambda'} (-)^{\lambda+\lambda'} (2\lambda'+1) \begin{Bmatrix} j_1 & j_3 & \lambda' \\ j_4 & j_2 & \lambda \end{Bmatrix} \langle (j_1 \bar{j}_3)_{\lambda'} | V^{a\lambda' R} | (j_2 \bar{j}_4)_{\lambda'} \rangle \end{aligned} \qquad 1.14$$

and anti-symmetric matrix elements can be constructed as

$$\begin{aligned} \langle (j_1 \bar{j}_2)_\lambda | V^{a\lambda R} | (j_3 \bar{j}_4)_\lambda \rangle = {} & \langle (j_1 \bar{j}_2)_\lambda | V^{\lambda R} | (j_3 \bar{j}_4)_\lambda \rangle + \\ & \sum_{\lambda'} (-)^{\lambda+\lambda'} (2\lambda'+1) \begin{Bmatrix} j_1 & j_2 & \lambda \\ j_4 & j_3 & \lambda' \end{Bmatrix} \langle (j_1 \bar{j}_3)_{\lambda'} | V^{\lambda' R} | (j_2 \bar{j}_4)_{\lambda'} \rangle \end{aligned} \qquad 1.15$$

Also, the relation between (pp)-coupling and (ph)-coupling is simplified to

$$\langle (j_1 j_4)_L | V^{aLR} | (j_2 j_3)_L \rangle = (-)^{L+1} \sum_{\lambda} (2\lambda+1) \begin{Bmatrix} j_1 & j_2 & \lambda \\ j_3 & j_4 & L \end{Bmatrix} \langle (j_1 \bar{j}_2)_\lambda | V^{a\lambda R} | (j_3 \bar{j}_4)_\lambda \rangle \qquad 1.16$$

or it's reverse relation

$$\langle (j_1 \bar{j}_2)_\lambda | V^{a\lambda R} | (j_3 \bar{j}_4)_\lambda \rangle = \sum_{L} (-)^{L+1} (2L+1) \begin{Bmatrix} j_1 & j_2 & \lambda \\ j_3 & j_4 & L \end{Bmatrix} \langle (j_1 j_4)_L | V^{aLR} | (j_2 j_3)_L \rangle \qquad 1.17$$

In all the nuclear structure calculations we will use anti-symmetric matrix elements that include the "Ring"-phase. For simplicity we will drop the superscripts aR throughout.

2. General considerations and relationships

We assume the interaction can be written as a scalar product of tensors of rank (k). Then the ph-ph matrix element including the "Ring"-phase of $(-)^{(k_1+k_3)}$ can be computed by using (Ed.7.1.6) as

$$\begin{aligned}
< (1,\bar{2})_\lambda|V^R|(3,\bar{4})_\lambda > =& (-)^{(k_1+k_3)} < (1,\bar{2})_\lambda|V|(3,\bar{4})_\lambda > = \\
=& < (1,\bar{2})_\lambda|\left[U^{R(k)}(1)\odot V^{R(k)}(2)\right]|(3,\bar{4})_\lambda > = \\
=& \delta_{k,\lambda}(-)^{k_1}\frac{1}{\hat{\lambda}} < 1\|U^{(\lambda)}\|2 > \quad (-)^{k_4}\frac{1}{\hat{\lambda}} < 4\|V^{(\lambda)}\|3 >
\end{aligned} \tag{2.1}$$

Thus it is necessary to bring the various interactions into this form. Note: All reduced one-body matrix elements should contain an additional factor of $\sqrt{\frac{1}{4\pi}}$. We have omitted this factor consistently and instead applied it to the radial integrals.

We will use the Fourier transform to separate the variables r_1 and r_2. For this the general relation can be worked out :

$$\begin{aligned}
V(r_{12})Y_{J,M}(\hat{r}_{12}) =& \frac{2}{\pi}\int q^2 dq \tilde{V}^J(q)\sum_{\ell_1,\ell_2}\frac{\hat{\ell}_1\hat{\ell}_2}{\hat{J}}\frac{1}{\sqrt{4\pi}}\langle \ell_1 0 \ell_2 0|J0\rangle \\
&\times j_{\ell_1}(qr_1)j_{\ell_2}(qr_2)(i)^{(\ell_1-\ell_2-J)}\left[Y_{\ell_1}(\hat{r}_1)\otimes Y_{\ell_2}(\hat{r}_2)\right]^{(J,M)}
\end{aligned} \tag{2.2}$$

where the form factor of the interaction is

$$\tilde{V}^J(q) = 4\pi\int V(r)j_J(qr)r^2 dr \tag{2.3}$$

The operators are defined as

$$\begin{aligned}
\vec{S} &= \frac{1}{2}(\vec{\sigma}_1+\vec{\sigma}_2) \\
\vec{r}_{12} &= \vec{r}_1 - \vec{r}_2 \\
\vec{L} &= \frac{1}{(2i)}\vec{r}_{12}\times(\vec{\nabla}_1-\vec{\nabla}_2)
\end{aligned} \tag{2.4}$$

Using (Ed.5.1.4)

$$(\hat{r}_{12})^{(1)} = \sqrt{\frac{4\pi}{3}}Y^{(1)}(\hat{r}_{12}) \tag{2.5}$$

and (Ed.5.2.4)

$$a^{(k)}\odot b^{(k)} = (-)^k\hat{k}\left[a^{(k)}\otimes b^{(k)}\right]^{(0)} \tag{2.6}$$

Using (Ed.5.1.8) we write

$$\begin{aligned}
L^{(1)} =& \frac{-1}{\sqrt{2}}r_{12}\left[(\hat{r}_{12})^{(1)}\otimes(\vec{\nabla}_1-\vec{\nabla}_2)^{(1)}\right]^{(1)} \\
=& -r_{12}\sqrt{\frac{2\pi}{3}}\left[Y^{(1)}(\hat{r}_{12})\otimes(\vec{\nabla}_1-\vec{\nabla}_2)^{(1)}\right]^{(1)}
\end{aligned} \tag{2.7}$$

For terms containing L^2 we need to evaluate the commutator for the tensor components with $\vec{\nabla} = \vec{\nabla}_1 - \vec{\nabla}_2$ which takes the form

$$\left[\vec{r}^{(1)}_s, \vec{\nabla}^{(1)}_{-t}\right] = 2\delta_{s,t}$$

with this we obtain

$$\begin{aligned}
\left[\left[\vec{r}\otimes\vec{\nabla}\right]^{(1)}\otimes\left[\vec{r}\otimes\vec{\nabla}\right]^{(1)}\right]^{(j)} =& 3\sum_{J_r,J_p}\hat{J}_r\hat{J}_p\begin{Bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ J_r & J_p & j\end{Bmatrix}\left[\left[\vec{r}\otimes\vec{r}\right]^{(J_r)}\otimes\left[\vec{\nabla}\otimes\vec{\nabla}\right]^{(J_p)}\right]^{(j)} \\
&+6(-)^{(j+1)}\begin{Bmatrix} 1 & 1 & 1 \\ 1 & 1 & j\end{Bmatrix}\left[\vec{r}\otimes\vec{\nabla}\right]^{(j)}
\end{aligned} \tag{2.8}$$

Due to vanishing $\vec{a} \times \vec{a}$, each of j, J_r, J_p can take up only the values of 0,2. For this operator the following relation is useful

$$\left[\vec{r} \otimes \vec{r}\right]^{(J)} = r^2 \frac{4\pi}{3} \left[Y^{(1)}(\hat{r}) \otimes Y^{(1)}(\hat{r})\right]^{(J)} = r^2 \sqrt{\frac{4\pi}{2J+1}} < 1010|J0 > Y^{(J)}(\hat{r}) \qquad 2.9$$

Thus we can write

$$\begin{aligned}\left[\left[\vec{r} \otimes \vec{\nabla}\right]^{(1)} \otimes \left[\vec{r} \otimes \vec{\nabla}\right]^{(1)}\right]^{(j)} =& 3r^2 \sum_{J_r, J_p} \hat{J}_p \begin{Bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ J_r & J_p & j \end{Bmatrix} \langle 1\ 0\ 1\ 0|J_r\ 0\rangle \sqrt{4\pi} \left[Y^{(J_r)}(\hat{r}) \otimes \left[\vec{\nabla} \otimes \vec{\nabla}\right]^{(J_p)}\right]^{(j)} \\ & - 6r \begin{Bmatrix} 1 & 1 & 1 \\ 1 & 1 & j \end{Bmatrix} \sqrt{\frac{4\pi}{3}} \left[Y^{(1)}(\hat{r}) \otimes \vec{\nabla}\right]^{(j)}\end{aligned} \qquad 2.10$$

3. Central interaction

Matrix elements for the interaction term $V^c(r_{12})$. This term includes the Coulomb interaction. The matrix element contributes only for natural parity matrix elements, i.e. $\lambda + l_1 + l_2 = even$. Including the "Ring"-phase of $(-)^{k_1+k_3}$ we get

$$\langle (1\bar{2})_\lambda | V^d | (3\bar{4})_\lambda \rangle = \frac{1}{2\lambda+1} (-)^{(k_1+k_4)} \frac{2}{\pi} \int q^2 dq\ F^c(q) \left(\langle 1 \| Y_\lambda \| 2 \rangle \langle 4 \| Y_\lambda \| 3 \rangle \right) \qquad 3.1$$

where the form factor for the central interaction is

$$F^c(q) = 4\pi \int V^c(r_{12})\ j_0(qr_{12})\ r_{12}^2 dr_{12} \qquad 3.2$$

Thus for the evaluation of the matrix element we need the following reduced matrix elements in which we include the radial matrix elements involving the Bessel functions: These also contain the "Ring"-phase. They are evaluated for terms that do not contain σ using (Ed.7.1.7)

$$\frac{1}{\hat{k}} (-)^{k_1} < (\ell_1 s_1) j_1 \| G^{(k)} \| (\ell_2 s_2) j_2 >= (-)^{(\ell_1+k_1+k_2+k)} \frac{\hat{j}_1 \hat{j}_2}{\hat{k}} \begin{Bmatrix} \ell_1 & \ell_2 & k \\ j_2 & j_1 & 1/2 \end{Bmatrix} < \ell_1 \| G^{(k)} \| \ell_2 > \qquad 3.3$$

We abbreviate this as

$$f(k, 1, 2) < \ell_1 \| G^{(k)} \| \ell_2 > \qquad 3.4$$

At this point we need the following reduced matrix elements:

$$< \ell_1 \| Y^\ell \| \ell_2 >= (-)^{\ell_2} \hat{\ell}_1 \hat{\ell}_2 < \ell_1 0 \ell_2 0 | \ell 0 > \langle R_1(r) | j_\lambda(qr) | R_2(r) \rangle \qquad 3.5$$

Ed.(5.4.5).We abbreviate this as

$$y(\ell, 1, 2) \langle R_1(r) | j_\lambda(qr) | R_2(r) \rangle \qquad 3.6$$

With these definitions we write the reduced matrix elements as

$$\frac{1}{\hat{\lambda}} (-)^{k_1} \langle 1 \| Y^\lambda \| 2 \rangle = f(\lambda, 1, 2) y(\lambda, 1, 2) \langle R_1(r) | j_\lambda(qr) | R_2(r) \rangle \qquad 3.7$$

4. Term (LL)

We write the interaction using eq.(0.8) with $j = 0$ as

$$\begin{aligned}
V =& V^{L2}(r_{12})\big[L \odot L\big] = -\sqrt{3}V^{L2}(r_{12})\big[L^{(1)} \otimes L^{(1)}\big]^{(0)} \\
&= -\frac{\sqrt{3}}{2}V^{L2}(r_{12})\Big[\big[\vec{r}_{12} \otimes \vec{\nabla}_{12}\big]^{(1)} \otimes \big[\vec{r}_{12} \otimes \vec{\nabla}_{12}\big]^{(1)}\Big]^{(0)} \\
&= -\frac{\sqrt{3}}{2}r_{12}^2 V^{L2}(r_{12}) \sum_{J_r,J_p} \langle 1010|J_r 0\rangle \sqrt{4\pi}\; 3\hat{J}_p \begin{Bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ J_r & J_p & 0 \end{Bmatrix} \Big[Y^{(J_r)}(\hat{r}_{12}) \otimes \big[\vec{\nabla}_{12} \otimes \vec{\nabla}_{12}\big]^{(J_p)}\Big]^{(0)} \\
&\quad + \frac{6}{2}\sqrt{4\pi}\, \begin{Bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{Bmatrix} r_{12} V^{L2}(r_{12})\big[Y^{(1)}(\hat{r}_{12}) \otimes \vec{\nabla}_{12}\big]^{(0)} \\
&= \frac{3}{2}\sqrt{4\pi}\; r_{12}^2 V^{L2}(r_{12}) \sum_J \langle 1010|J0\rangle \begin{Bmatrix} 1 & 1 & 1 \\ 1 & 1 & J \end{Bmatrix} \Big[Y^{(J)}(\hat{r}_{12}) \otimes \big[\vec{\nabla}_{12} \otimes \vec{\nabla}_{12}\big]^{(J)}\Big]^{(0)} \\
&\quad - \sqrt{4\pi}\; r_{12} V^{L2}(r_{12})\big[Y^{(1)}(\hat{r}_{12}) \otimes \vec{\nabla}_{12}\big]^{(0)}
\end{aligned} \qquad 4.1$$

As the $9j$-coefficient requires $J_r = J_p$ we have set $J_r = J_p = J$. J can assume the values of 0 or 2. We employ the Fourier transform where we define the form factors of this interaction as

$$\tilde{V}^{L2,J}(q) = 4\pi \int r_{12}^4 V^{L2}(r_{12}) j_J(qr_{12}) dr_{12} \qquad 4.2$$

and similarly

$$\bar{V}^{L2,1}(q) = 4\pi \int r_{12}^3 V^{L2}(r_{12}) j_1(qr_{12}) dr_{12} \qquad 4.3$$

Then, using eq.(2.2) with eq.(4.1) we get

$$\begin{aligned}
V =& \sum_J \frac{3}{2}\langle 1010|J0\rangle \frac{1}{\hat{J}} \begin{Bmatrix} 1 & 1 & 1 \\ 1 & 1 & J \end{Bmatrix} \sum_{\ell_1,\ell_2} (-)^{(\ell_1+\ell_2+J)/2} \hat{\ell}_1 \hat{\ell}_2 < \ell_1 0 \ell_2 0|J0 > \frac{2}{\pi} \int q^2 dq \tilde{V}^{L2J}(q) j_{\ell_1}(qr_1) j_{\ell_2}(qr_2) \\
&\times (-)^{\ell_2} \Big[\big[Y^{(\ell_1)}(\hat{r}_1) \otimes Y^{(\ell_2)}(\hat{r}_2)\big]^{(J)} \otimes \big[\vec{\nabla}_{12} \otimes \vec{\nabla}_{12}\big]^{(J)}\Big]^{(0)} \\
&+ \sum_{\ell_1,\ell_2} (-)^{(\ell_1+\ell_2+1)/2} \frac{\hat{\ell}_1 \hat{\ell}_2}{\sqrt{3}} < \ell_1 0 \ell_2 0|10 > \frac{2}{\pi} \int q^2 dq \bar{V}^{L2}(q) j_{\ell_1}(qr_1) j_{\ell_2}(qr_2) \\
&\times (-)^{\ell_2} \Big[\big[Y^{(\ell_1)}(\hat{r}_1) \otimes Y^{(\ell_2)}(\hat{r}_2)\big]^{(1)} \otimes \vec{\nabla}_{12}\Big]^{(0)}
\end{aligned} \qquad 4.4$$

Substituting $\vec{\nabla}_{12} = \vec{\nabla}_1 - \vec{\nabla}_2$ results in five terms which need to be recoupled in order to be of the form $F^{(k)}(1) \odot G^{(k)}(2)$. These are

$$\begin{aligned}
Z_1 =& (-)^{\ell_2} \Big[\big[Y^{(\ell_1)}(1) \otimes Y^{(\ell_2)}(2)\big]^{(J)} \otimes \big[\vec{\nabla}_1 \otimes \vec{\nabla}_1\big]^{(J)}\Big]^{(0)} \\
=& \frac{1}{\hat{\ell}_2} \Big[\Big[Y^{(\ell_1)}(1) \otimes \big[\vec{\nabla}_1 \otimes \vec{\nabla}_1\big]^{(J)}\Big]^{(\ell_2)} \odot Y^{(\ell_2)}(2)\Big] \\
Z_2 =& (-2)(-)^{\ell_2} \Big[\big[Y^{(\ell_1)}(1) \otimes Y^{(\ell_2)}(2)\big]^{(J)} \otimes \big[\vec{\nabla}_1 \otimes \vec{\nabla}_2\big]^{(J)}\Big]^{(0)} \\
=& 2\sum_k \hat{J} \begin{Bmatrix} \ell_1 & \ell_2 & J \\ 1 & 1 & k \end{Bmatrix} \Big[\big[Y^{(\ell_1)}(1) \otimes \vec{\nabla}_1\big]^{(k)} \odot \big[Y^{(\ell_2)}(2) \otimes \vec{\nabla}_2\big]^{(k)}\Big]
\end{aligned}$$

$$
\begin{aligned}
Z_3 =&(-)^{\ell_2}\Big[\big[Y^{(\ell_1)}(1)\otimes Y^{(\ell_2)}(2)\big]^{(J)}\otimes\big[\vec{\nabla}_2\otimes\vec{\nabla}_2\big]^{(J)}\Big]^{(0)}\\
=&\frac{1}{\hat{\ell}_1}\Big[Y^{(\ell_1)}(1)\odot\Big[Y^{(\ell_2)}(2)\otimes\big[\vec{\nabla}_2\otimes\vec{\nabla}_2\big]^{(J)}\Big]^{(\ell_1)}\Big]\\
Z_4 =&(-)^{\ell_2}\Big[\big[Y^{(\ell_1)}(1)\otimes Y^{(\ell_2)}(2)\big]^{(1)}\otimes\vec{\nabla}_1\Big]^{(0)}\\
=&\frac{1}{\hat{\ell}_2}\Big[\big[Y^{(\ell_1)}(1)\otimes\vec{\nabla}_1\big]^{(\ell_2)}\odot Y^{(\ell_2)}(2)\Big]\\
Z_5 =&-(-)^{\ell_2}\Big[\big[Y^{(\ell_1)}(1)\otimes Y^{(\ell_2)}(2)\big]^{(1)}\otimes\vec{\nabla}_2\Big]^{(0)}\\
=&\frac{1}{\hat{\ell}_1}\Big[Y^{(\ell_1)}(1)\odot\big[Y^{(\ell_2)}(2)\otimes\vec{\nabla}_2\big]^{(\ell_1)}\Big]
\end{aligned}
\qquad 4.5
$$

For the evaluation of this matrix element we need the additional reduced matrix elements: The matrix element involving the operator ∇ is

$$
\begin{aligned}
&<\ell_1\|(Y^\ell\nabla)^{(\kappa)}\|\ell_2>=(-)^{\ell_2+\kappa}\hat{\ell}_1\hat{\ell}\hat{\kappa}\\
&\Bigg\{\sqrt{(2\ell_2+3)(\ell_2+1)}\begin{Bmatrix}\ell & 1 & \kappa\\ \ell_2 & \ell_1 & \ell_2+1\end{Bmatrix}\begin{pmatrix}\ell_1 & \ell & \ell_2+1\\ 0 & 0 & 0\end{pmatrix}\langle R_1(r)|j_\ell(qr)(\frac{d}{dr}-\frac{\ell_2}{r})|R_2(r)\rangle\\
&\quad-\sqrt{(2\ell_2-1)\ell_2}\begin{Bmatrix}\ell & 1 & \kappa\\ \ell_2 & \ell_1 & \ell_2-1\end{Bmatrix}\begin{pmatrix}\ell_1 & \ell & \ell_2-1\\ 0 & 0 & 0\end{pmatrix}\langle R_1(r)|j_\ell(qr)(\frac{d}{dr}+\frac{\ell_2+1}{r})|R_2(r)\rangle\Bigg\}
\end{aligned}
\qquad 4.6
$$

which we abbreviate as

$$
\sum_{i=1,2} del(i,\ell,\kappa,1,2)\langle R_1(r)|j_\ell(qr)\ op(i)|R_2(r)\rangle \qquad 4.7
$$

where $op(1)=\frac{d}{dr}$ and $op(2)=\frac{1}{r}$. And

$$
\begin{aligned}
&\langle\ell_1\|\Big[Y^{(\ell)}\otimes\big[\vec{\nabla}\otimes\vec{\nabla}\big]^{(J)}\Big]^{(\lambda)}\|\ell_2\rangle=(-)^{(\ell_2+\lambda+J)}\hat{\lambda}\hat{J}\hat{\ell}_1\hat{\ell}\\
&\Big[\begin{Bmatrix}\ell & J & \lambda\\ \ell_2 & \ell_1 & \ell_2+2\end{Bmatrix}\begin{Bmatrix}1 & 1 & J\\ \ell_2 & \ell_2+2 & \ell_2+1\end{Bmatrix}\begin{pmatrix}\ell_1 & \ell & \ell_2+2\\ 0 & 0 & 0\end{pmatrix}\\
&\qquad\times(\widehat{\ell_2+2})\sqrt{\ell_2+1}\sqrt{\ell_2+2}<R_1(r)|j_\ell(qr)(\frac{d}{dr}-\frac{\ell_2+1}{r})(\frac{d}{dr}-\frac{\ell_2}{r})|R_2(r)>\\
&-\begin{Bmatrix}\ell & J & \lambda\\ \ell_2 & \ell_1 & \ell_2\end{Bmatrix}\begin{Bmatrix}1 & 1 & J\\ \ell_2 & \ell_2 & \ell_2+1\end{Bmatrix}\begin{pmatrix}\ell_1 & \ell & \ell_2\\ 0 & 0 & 0\end{pmatrix}\\
&\qquad\times\hat{\ell}_2(\ell_2+1)<R_1(r)|j_\ell(qr)(\frac{d}{dr}+\frac{\ell_2+2}{r})(\frac{d}{dr}-\frac{\ell_2}{r})|R_2(r)>\\
&-\begin{Bmatrix}\ell & J & \lambda\\ \ell_2 & \ell_1 & \ell_2\end{Bmatrix}\begin{Bmatrix}1 & 1 & J\\ \ell_2 & \ell_2 & \ell_2-1\end{Bmatrix}\begin{pmatrix}\ell_1 & \ell & \ell_2\\ 0 & 0 & 0\end{pmatrix}\\
&\qquad\times\hat{\ell}_2\ell_2<R_1(r)|j_\ell(qr)(\frac{d}{dr}-\frac{\ell_2-1}{r})(\frac{d}{dr}+\frac{\ell_2+1}{r})|R_2(r)>\\
&+\begin{Bmatrix}\ell & J & \lambda\\ \ell_2 & \ell_1 & \ell_2-2\end{Bmatrix}\begin{Bmatrix}1 & 1 & J\\ \ell_2 & \ell_2-2 & \ell_2-1\end{Bmatrix}\begin{pmatrix}\ell_1 & \ell & \ell_2-2\\ 0 & 0 & 0\end{pmatrix}\\
&\qquad\times(\widehat{\ell_2-2})\sqrt{\ell_2-1}\sqrt{\ell_2}<R_1(r)|j_\ell(qr)(\frac{d}{dr}+\frac{\ell_2}{r})(\frac{d}{dr}+\frac{\ell_2+1}{r})|R_2(r)>
\end{aligned}
\qquad 4.8
$$

Here we give the reduced matrix elements separately for $J=0$ and for $J=2$ since they combine with different form factors. For $J=0$ the selection rules require $\ell=\lambda$. Thus for $J=0$ we can abbreviate (4.8) as

$$
\sum_{i=1,3} ddel_0(i,\lambda,1,2)<R_1(r)|j_\ell(qr)op'(i)|R_2(r)> \qquad 4.9
$$

whereas for $J = 2$ we abbreviate this as

$$\sum_{i=1,3} ddel_2(i, \ell, \lambda, 1, 2) < R_1(r)|j_\ell(qr) op'(i)|R_2(r) > \qquad 4.10$$

with $op'(1) = \frac{d^2}{dr^2}$, $op'(2) = \frac{1}{r}\frac{d}{dr}$, and $op'(3) = \frac{1}{r^2}$.

Furthermore, we note that

$$\frac{3}{2}\langle 1010|00\rangle \left\{ \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 0 \end{array} \right\} = \frac{1}{\sqrt{12}} \qquad 4.11$$

and

$$\frac{3}{2}\langle 1010|20\rangle \left\{ \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 2 \end{array} \right\} \frac{1}{\sqrt{5}} = \sqrt{\frac{1}{120}} \qquad 4.12$$

so that the matrix element can be written as

$$\begin{aligned} < (1\bar{2})_\lambda | V^{R,L2}_{(J=0)} | (3\bar{4})_\lambda > &= \frac{1}{2\lambda+1} (-)^{k_1+k_4} \frac{2}{\pi} \int q^2 dq \tilde{V}^{L20}(q) \\ & \frac{1}{\sqrt{12}} \Big(< 1 \| \big(Y^\lambda \otimes \left[\nabla \otimes \nabla \right]^{(0)} \big)^{(\lambda)} \| 2 >< 4 \| Y^\lambda \| 3 > \\ & + < 1 \| Y^\lambda \| 2 >< 4 \| \big(Y^\lambda \otimes \left[\nabla \otimes \nabla \right]^{(0)} \big)^{(\lambda)} \| 3 > \Big) \end{aligned} \qquad 4.13$$

This matrix element contributes only in natural parity cases.

$$\begin{aligned} < (1\bar{2})_\lambda | V^{R,L2}_{Z=2,(J=0)} | (3\bar{4})_\lambda >= & \frac{1}{2\lambda+1} (-)^{k_1+k_4} \frac{2}{\pi} \int q^2 dq \tilde{V}^{L20}(q) \\ & \sum_{\ell=\lambda-1,.,\lambda+1} (-)^{\ell+\lambda+1} \frac{1}{3} < 1 \| (Y^\ell \nabla)^\lambda \| 2 >< 4 \| (Y^\ell \nabla)^\lambda \| 3 > \Big) \end{aligned} \qquad 4.14$$

For the term with $J = 1$, linear in r, the contributions are from Z_4 and Z_5:

$$\begin{aligned} & < (1\bar{2})_\lambda | V^{R,L2}_{(J=1)} | (3\bar{4})_\lambda >= \\ & \frac{1}{\sqrt{3}} \frac{1}{2\lambda+1} (-)^{(k_1+k_4)} \sum_{\ell=\lambda-1,\lambda+1} (-)^{(\ell+\lambda+1)/2} \hat{\ell} < \ell 0 \lambda 0 | 10 > \frac{2}{\pi} \int q^2 dq \bar{V}^{L2,1}(q) \\ & \Big(< 1 \| (Y^\ell \nabla)^{(\lambda)} \| 2 >< 4 \| Y^\lambda \| 3 > \\ & + < 1 \| Y^\lambda \| 2 >< 4 \| (Y^\ell \nabla)^{(\lambda)} \| 3 > \Big) \end{aligned} \qquad 4.15$$

This term contributes only in natural parity cases. We have to consider $\ell = \lambda - 1$ and $\ell = \lambda + 1$. Finally, the contributions from $J = 2$ can be written

$$\begin{aligned} < (1\bar{2})_\lambda | V^{R,L2}_{(J=2)Z=1} | (3\bar{4})_\lambda >= & -\frac{1}{\sqrt{120}} \frac{1}{2\lambda+1} (-)^{(k_1+k_4)} \frac{2}{\pi} \int q^2 dq \tilde{V}^{L2,2}(q) \\ \sum_{\ell=\lambda-2,\lambda,\lambda+2} (-)^{(\ell+\lambda)/2} \hat{\ell} < \ell 0 \lambda 0 | 20 > & \Big(< 1 \| \left[Y^\ell (\nabla \otimes \nabla)^{(2)} \right]^{(\lambda)} \| 2 >< 4 \| Y^\lambda \| 3 > \\ & + < 1 \| Y^\lambda \| 2 >< 4 \| \left[Y^\ell (\nabla \otimes \nabla)^{(2)} \right]^{(\lambda)} \| 3 > \Big) \end{aligned} \qquad 4.16$$

Again, this term contributes only in natural parity cases with $\ell = \lambda - 2$, $\ell = \lambda$, and $\ell = \lambda + 2$. with

$$d_{lsq}(\ell_a, \ell_b, \lambda) = \sqrt{\frac{1}{6}} (-)^{(\ell_a+\ell_b)/2} \hat{\ell}_a \hat{\ell}_b < \ell_a 0 \ell_b 0 | 20 > \left\{ \begin{array}{ccc} 1 & 1 & 2 \\ \ell_a & \ell_b & \lambda \end{array} \right\} \qquad 4.17$$

the remaining contribution is

$$
\begin{aligned}
&< (1\bar{2})_\lambda | V^{R,L2}_{(J=2)Z=2} | (3\bar{4})_\lambda >= \\
&\quad -\frac{1}{2\lambda+1}(-)^{(k_1+k_4)}\frac{2}{\pi}\int q^2 dq \tilde{V}^{L2,2}(q) \\
&\quad \sum_{\ell_a=\lambda\pm1}\sum_{\ell_b=\lambda\pm1} d_{lsq}(\ell_a,\ell_b,\lambda) < 1\|(Y^{\ell_a}\nabla)^{(\lambda)}\|2 >< 4\|(Y^{\ell_b}\nabla)^{(\lambda)}\|3 >\Big)
\end{aligned} \qquad 4.18
$$

This term contributes in both, natural and unnatural parity cases. We have to consider the three cases with $\ell_a = \ell_b = \lambda - 1,\ \lambda$, and $\lambda + 1$ as well as the two cases: $\ell_a = \lambda - 1, \ell_b = \lambda + 1$ and $\ell_a = \lambda + 1, \ell_b = \lambda - 1$. With our abbreviations the reduced matrix elements can be written as

$$
(-)^{k_1}\frac{1}{\hat{\lambda}}\langle 1\|(Y^\ell\vec{\nabla})^\lambda\|2\rangle = \sum_{i=1,2} f(\lambda,1,2)del(i,\ell,\lambda,1,2)\langle R_1(r)|j_\ell(qr)\ op(i)|R_2(r)\rangle \qquad 4.19
$$

$$
\frac{1}{\hat{\lambda}}(-)^{k_1} < 1\|\big[Y^\lambda(\nabla\otimes\nabla)^{(0)}\big]^{(\lambda)}\|2 >= \sum_{i=1,3} f(\lambda,1,2)ddel_0(i,\lambda,1,2)\langle R_1(r)|j_\ell(qr)\ op'(i)|R_2(r)\rangle \qquad 4.20
$$

and

$$
\frac{1}{\hat{\lambda}}(-)^{k_1} < 1\|\big[Y^\ell(\nabla\otimes\nabla)^{(2)}\big]^{(\lambda)}\|2 >= \sum_{i=1,3} f(\lambda,1,2)ddel_2(i,\ell,\lambda,1,2)\langle R_1(r)|j_\ell(qr)\ op'(i)|R_2(r)\rangle \qquad 4.21
$$

5. Terms $\sigma\sigma$ and (LL)$\sigma\sigma$

Such terms can be obtained from the previous terms such as the central interaction or the L^2-interaction by adding the $\sigma_1\sigma_2$ term. This results in

$$
V = V^{L2\sigma}(r_{12})\big[L\odot L\big]\big[\vec{\sigma}_1\odot\vec{\sigma}_2\big] \qquad 5.1
$$

We can obtain the separated interaction immediately from the separated form for the L^2 term via

$$
\big[F^{(J)}(1)\odot G^{(J)}(2)\big]\big[\vec{\sigma}_1\odot\vec{\sigma}_2\big] = \sum_k (-)^{(k+J+1)}\Big[\big[F^{(J)}(1)\otimes\vec{\sigma}_1\big]^{(k)}\odot\big[G^{(J)}(2)\otimes\vec{\sigma}_2\big]^{(k)}\Big] \qquad 5.2
$$

Thus, we can immediately take the results of the previous sections and write the matrix element for the σ-interaction as

$$
\begin{aligned}
< (1\bar{2})_\lambda | V^{R,\sigma}_{(J=0)} | (3\bar{4})_\lambda >=& \frac{1}{2\lambda+1}(-)^{k_1+k_4}\frac{2}{\pi}\int q^2 dq \tilde{V}^\sigma(q) \\
&\sum_{\ell=\lambda-1,.,\lambda+1}(-)^{(\ell+\lambda+1)} < 1\|(Y^\ell\sigma)^\lambda\|2 >< 4\|(Y^\ell\sigma)^\lambda\|3 >
\end{aligned} \qquad 5.3
$$

Similarly we write the contributions to the matrix elements of the $L^2\sigma$ interaction as

$$
\begin{aligned}
< (1\bar{2})_\lambda | V^{R,L2\sigma}_{(J=0)} | (3\bar{4})_\lambda >=& \frac{1}{\sqrt{12}}\frac{1}{2\lambda+1}(-)^{k_1+k_4}\frac{2}{\pi}\int q^2 dq \tilde{V}^{L20\sigma}(q)\sum_{\ell=\lambda-1,.,\lambda+1}(-)^{(\ell+\lambda+1)} \\
&\Big(< 1\|\Big[\big(Y^\ell\otimes\big[\nabla\otimes\nabla\big]^{(0)}\big)^{(\ell)}\sigma\Big]^\lambda\|2 >< 4\|(Y^\ell\sigma)^\lambda\|3 > \\
&+ < 1\|(Y^\ell\sigma)^\lambda\|2 >< 4\|\Big[\big(Y^\ell\otimes\big[\nabla\otimes\nabla\big]^{(0)}\big)^\ell\sigma\Big]^{(\lambda)}\|3 >\Big)
\end{aligned} \qquad 5.4
$$

We include three cases: $\ell = \lambda - 1$, $\ell = \lambda$, and $\ell - \lambda + 1$.

$$< (1\bar{2})_\lambda | V^{R,L2\sigma}_{Z=2(J=0)} | (3\bar{4})_\lambda >= \frac{1}{3}\frac{1}{2\lambda+1}(-)^{k_1+k_4}\frac{2}{\pi}\int q^2 dq \tilde{V}^{L20\sigma}(q) \sum_\kappa \sum_{\ell=\kappa-1,\kappa+1} (-)^{(\ell+\lambda)} < 1\| \left[(Y^\ell\nabla)^\kappa\sigma\right]^{(\lambda)} \|2 >< 4\| \left[(Y^\ell\nabla)^\kappa\sigma\right]^{(\lambda)} \|3 > \Big\} \qquad 5.5$$

In this term ℓ can take the values of $\lambda-2$, $\lambda-1$, λ, $\lambda+1$, and $\lambda+2$. For the term with $J=1$, linear in r the contributions are from Z_4 and Z_5:

$$< (1\bar{2})_\lambda | V^{R,L2\sigma}_{(J=1)} | (3\bar{4})_\lambda >= \sum_\kappa \sum_{\ell=\kappa-1,\kappa+1} (-)^{(\ell+\kappa+1)/2}(-)^{(\kappa+\lambda+1)} \frac{1}{\sqrt{3}}\frac{1}{2\lambda+1}(-)^{(k_1+k_4)}\hat{\ell} < \ell 0 \kappa 0 | 10 > \frac{2}{\pi}\int q^2 dq \bar{V}^{L2\sigma,1}(q) \Big(< 1\| \left[(Y^\ell\nabla)^{(\kappa)}\sigma\right]^{(\lambda)} \|2 >< 4\| (Y^\kappa\sigma)^{(\lambda)} \|3 > + < 1\| (Y^\kappa\sigma)^{(\lambda)} \|2 >< 4\| \left[(Y^\ell\nabla)^{(\kappa)}\sigma\right]^{(\lambda)} \|3 > \Big) \qquad 5.6$$

Here κ can be $\lambda-1$, λ, and $\lambda+1$. For each case ℓ can be $\kappa+1$ and $\kappa-1$. Finally, the contributions from $J=2$ can be written

$$< (1\bar{2})_\lambda | V^{R,L2\sigma}_{(J=2)Z=1} | (3\bar{4})_\lambda >= \sum_\kappa (-)^{(\kappa+\lambda)} \frac{1}{\sqrt{120}}\frac{1}{2\lambda+1}(-)^{(k_1+k_4)}\frac{2}{\pi}\int q^2 dq \tilde{V}^{L2\sigma,2}(q) \sum_{\ell=\kappa-2,\kappa,\kappa+2} (-)^{(\ell+\kappa)/2)}\hat{\ell} < \ell 0 \kappa 0 | 20 > \Big(< 1\| \left[\left[Y^\ell(\nabla\otimes\nabla)^{(2)}\right]^{(\kappa)}\sigma\right]^{(\lambda)} \|2 >< 4\| (Y^\kappa\sigma)^{(\lambda)} \|3 > + < 1\| (Y^\kappa\sigma)^{(\lambda)} \|2 >< 4\| \left[\left[Y^\ell(\nabla\otimes\nabla)^{(2)}\right]^{(\kappa)}\sigma\right]^{(\lambda)} \|3 > \Big) \qquad 5.7$$

Again, κ can take the values $\lambda-1$, λ, and $\lambda+1$. For each case ℓ can be κ, $\kappa-2$, and $\kappa+2$.

$$< (1\bar{2})_\lambda | V^{R,L2\sigma}_{(J=2)Z=2} | (3\bar{4})_\lambda >= \sum_\kappa (-)^{(\kappa+\lambda)} \frac{1}{\sqrt{120}}\frac{1}{2\lambda+1}(-)^{(k_1+k_4)}\frac{2}{\pi}\int q^2 dq \tilde{V}^{L2\sigma,2}(q) + \sum_{\ell_a=\kappa\pm1}\sum_{\ell_b=\kappa\pm1} dl(\ell_a,\ell_b,\kappa) < 1\| \left[(Y^{\ell_a}\nabla)^{(\kappa)}\sigma\right]^{(\lambda)} \|2 >< 4\| \left[(Y^{\ell_b}\nabla)^{(\kappa)}\sigma\right]^{(\lambda)} \|3 > \Big) \qquad 5.8$$

For the reduced one-body matrix elements containing σ we use (Ed.7.1.5)

$$\frac{1}{\hat{\lambda}}(-)^{k_1} < (\ell_1 s_1) j_1 \| (G^{(\ell)}\sigma)^{(\lambda)} \| (\ell_2 s_2) j_2 >= (-)^{k_1}\hat{j}_1\hat{j}_2\sqrt{6}\begin{Bmatrix} \ell_1 & \ell_2 & \ell \\ 1/2 & 1/2 & 1 \\ j_1 & j_2 & \lambda \end{Bmatrix} < \ell_1 \| G^{(\ell)} \| \ell_2 > \qquad 5.9$$

we abbreviate this as

$$\sum_\ell g(\ell,k,1,2) < \ell_1 \| G^{(\ell)} \| \ell_2 > \qquad 5.10$$

With our abbreviations the reduced matrix elements can be written as

$$(-)^{k_1}\frac{1}{\hat{\lambda}}\langle 1 \| (Y^\ell\vec{\sigma})^\lambda \| 2\rangle = g(\ell,\lambda,1,2) y(\ell,1,2)\ \langle R_1(r) | j_\ell(qr) | R_2(r)\rangle \qquad 5.11$$

$$\frac{1}{\hat{\lambda}}(-)^{k_1}\langle 1 \| \left[[Y^\ell\otimes\vec{\nabla}]^\kappa\otimes\vec{\sigma}\right]^\lambda \| 2\rangle = \sum_{i=1,2} g(\kappa,\lambda,1,2) del(i,\ell,\kappa,1,2)\langle R_1(r) | j_\ell(qr)\ op(i) | R_2(r)\rangle \qquad 5.12$$

$$\frac{1}{\hat{\lambda}}(-)^{k_1} < 1\| \left[\left[Y^\kappa(\nabla\otimes\nabla)^{(0)}\right]^{(\kappa)}\sigma\right]^{(\lambda)} \|2 >= \sum_{i=1,3} g(\kappa,\lambda,1,2) ddel_0(i,\kappa,1,2)\langle R_1(r) | j_\ell(qr)\ op'(i) | R_2(r)\rangle \qquad 5.13$$

and

$$\frac{1}{\hat{\lambda}}(-)^{k_1} < 1\| \left[\left[Y^\ell(\nabla\otimes\nabla)^{(2)}\right]^{(\kappa)}\sigma\right]^{(\lambda)} \|2 >= \sum_{i=1,3} g(\kappa,\lambda,1,2) ddel_2(i,\ell,\kappa,1,2)\langle R_1(r) | j_\ell(qr)\ op'(i) | R_2(r)\rangle \qquad 5.14$$

6. Spin-orbit (LS)

$$\begin{aligned} V =&V^{LS}(r_{12})\big[L^{(1)}\odot S^{(1)}\big] = -\sqrt{3}V^{LS}(r_{12})\big[L^{(1)}\otimes S^{(1)}\big]^{(0)} \\ =&+r_{12}V^{LS}(r_{12})\sqrt{2\pi}\Big[\big[Y^{(1)}(\hat{r}_{12})\otimes(\vec{\nabla}_1-\vec{\nabla}_2)^{(1)}\big]^{(1)}\otimes S^{(1)}\Big]^{(0)} \end{aligned} \qquad 6.1$$

We introduce the form-factor of the LS interaction:

$$\tilde{V}^{LS}(q) = 4\pi\int r_{12}^3 V^{LS}(r_{12})j_1(qr_{12})dr_{12} \qquad 6.2$$

and write using eq.(2.2) with $J=1$

$$\begin{aligned} V =&\sqrt{2\pi}\frac{2}{\pi}\sum_{\ell_1,\ell_2}\int q^2dq\tilde{V}^{LS}(q)j_{\ell_1}(qr_1)j_{\ell_2}(qr_2) \\ &\times\hat{\ell}_1\hat{\ell}_2\frac{1}{\sqrt{3\cdot 4\pi}} < \ell_1 0\ell_2 0|10 > (i)^{\ell_1}(-i)^{\ell_2+1} \\ &\times\Big[\Big[\big[Y^{(\ell_1)}(\hat{r}_1)\otimes Y^{(\ell_2)}(\hat{r}_2)\big]^{(1)}\otimes(\vec{\nabla}_1-\vec{\nabla}_2)^{(1)}\Big]^{(1)}\otimes S^{(1)}\Big]^{(0)} \end{aligned} \qquad 6.3$$

Or, combining factors

$$\begin{aligned} V =&\sum_{\ell_1,\ell_2}(-)^{(\ell_1-\ell_2-1)/2}\sqrt{\frac{1}{24}}\frac{2}{\pi}\int q^2dq\tilde{V}^{LS}(q)j_{\ell_1}(qr_1)j_{\ell_2}(qr_2) \\ &\times\hat{\ell}_1\hat{\ell}_2 < \ell_1 0\ell_2 0|10 > \\ &\times\Big[\Big[\big[Y^{(\ell_1)}(\hat{r}_1)\otimes Y^{(\ell_2)}(\hat{r}_2)\big]^{(1)}\otimes(\vec{\nabla}_1-\vec{\nabla}_2)^{(1)}\Big]^{(1)}\otimes(\sigma_1+\sigma_2)^{(1)}\Big]^{(0)} \end{aligned} \qquad 6.4$$

Inserting S from above, results in four terms as represented in the last line of the previous equation. They must be recoupled and brought into the form $F^{(k)}(1)\odot G^{(k)}(2)$. These terms are:

$$\begin{aligned} Z_1 =&\Big[\Big[\big[Y^{\ell_1}(1)\otimes Y^{\ell_2}(2)\big]^{(1)}\otimes\vec{\nabla}_1^{(1)}\Big]^{(1)}\otimes\sigma_1^{(1)}\Big]^{(0)} \\ =&-\sum_k\sqrt{3}\hat{k}\begin{Bmatrix}\ell_2 & \ell_1 & 1\\ 1 & 1 & k\end{Bmatrix}\Big[Y^{\ell_2}(2)\otimes\Big[\big[Y^{\ell_1}(1)\otimes\vec{\nabla}_1\big]^{(k)}\otimes\sigma_1\Big]^{(\ell_2)}\Big]^{(0)} \\ =&(-)^{(\ell_2+1)}\sum_k\sqrt{3}\frac{\hat{k}}{\hat{\ell}_2}\begin{Bmatrix}\ell_2 & \ell_1 & 1\\ 1 & 1 & k\end{Bmatrix}\Big[Y^{\ell_2}(2)\odot\Big[\big[Y^{\ell_1}(1)\otimes\vec{\nabla}_1\big]^{(k)}\otimes\sigma_1\Big]^{(\ell_2)}\Big] \\ Z_2 =&(-)\Big[\Big[\big[Y^{\ell_1}(1)\otimes Y^{\ell_2}(2)\big]^{(1)}\otimes\vec{\nabla}_2^{(1)}\Big]^{(1)}\otimes\sigma_1^{(1)}\Big]^{(0)} \\ =&(-)^{\ell_2}\sqrt{3}\sum_k\begin{Bmatrix}\ell_2 & \ell_1 & 1\\ 1 & 1 & k\end{Bmatrix}\Big[\big[Y^{\ell_1}(1)\otimes\sigma_1\big]^{(k)}\odot\big[Y^{\ell_2}(2)\otimes\nabla_2\big]^{(k)}\Big] \\ Z_3 =&\Big[\Big[\big[Y^{\ell_1}(1)\otimes Y^{\ell_2}(2)\big]^{(1)}\otimes\vec{\nabla}_1^{(1)}\Big]^{(1)}\otimes\sigma_2^{(1)}\Big]^{(0)} \\ =&(-)^{\ell_2}\sqrt{3}\sum_k\begin{Bmatrix}\ell_2 & \ell_1 & 1\\ 1 & 1 & k\end{Bmatrix}\Big[\big[Y^{\ell_2}(2)\otimes\sigma_2\big]^{(k)}\odot\big[Y^{\ell_1}(1)\otimes\nabla_1)\big]^{(k)}\Big] \\ Z_4 =&(-)\Big[\Big[\big[Y^{\ell_1}(1)\otimes Y^{\ell_2}(2)\big]^{(1)}\otimes\vec{\nabla}_2^{(1)}\Big]^{(1)}\otimes\sigma_2^{(1)}\Big]^{(0)} \\ =&(-)^{(\ell_2+1)}\sum_k\sqrt{3}\frac{\hat{k}}{\hat{\ell}_1}\begin{Bmatrix}\ell_2 & \ell_1 & 1\\ 1 & 1 & k\end{Bmatrix}\Big[Y^{\ell_1}(1)\odot\Big[\big[Y^{\ell_2}(2)\otimes\vec{\nabla}_2\big]^{(k)}\otimes\sigma_2\Big]^{(\ell_1)}\Big] \end{aligned} \qquad 6.5$$

by introducing

$$d_{so}(\ell_1,\ell_2,\lambda) = \frac{1}{\sqrt{8}}\hat{\ell}_1\hat{\lambda} < \ell_1\ 0\ \ell_2\ 0|1\ 0 > (-)^{(\ell_1+\ell_2+1)/2}\begin{Bmatrix} 1 & 1 & 1 \\ \ell_1 & \ell_2 & \lambda \end{Bmatrix} \qquad 6.6$$

we write the matrix element as

$$\begin{aligned} < (1,\bar{2})_\lambda|V^{R,LS}|(3,\bar{4})_\lambda > =& \frac{1}{2\lambda+1}(-)^{(k_1+k_4)}\sum_{\ell=\lambda\pm1}\frac{2}{\pi}\int q^2dq\ \tilde{V}^{LS}(q) \\ &\Bigg(\sum_k d_{so}(\ell,\lambda,k) < 1\|\Big[\big[Y^\ell\otimes\vec{\nabla}\big]^{(k)}\otimes\sigma\Big]^{(\lambda)}\|2 >< 4\|Y^\lambda\|3 > \\ &+\sum_k d_{so}(\ell,\lambda,k) < 4\|\Big[\big[Y^\ell\otimes\vec{\nabla}\big]^{(k)}\otimes\sigma\Big]^{(\lambda)}\|3 >< 1\|Y^\lambda\|2 > \\ &-d_{so}(\ell,\lambda,\lambda) < 1\|(Y^\ell\sigma)^{(\lambda)}\|2 >< 4\|(Y^\lambda\nabla)^{(\lambda)}\|3 > \\ &-d_{so}(\ell,\lambda,\lambda) < 4\|(Y^\ell\sigma)^{(\lambda)}\|3 >< 1\|(Y^\lambda\nabla)^{(\lambda)}\|2 > \\ &-d_{so}(\ell,\lambda,\lambda) < 1\|(Y^\lambda\sigma)^{(\lambda)}\|2 >< 4\|(Y^\ell\nabla)^{(\lambda)}\|3 > \\ &-d_{so}(\ell,\lambda,\lambda) < 4\|(Y^\lambda\sigma)^{(\lambda)}\|3 >< 1\|(Y^\ell\nabla)^{(\lambda)}\|2 >\Bigg) \end{aligned} \qquad 6.7$$

All the reduced one body matrix elements have been defined before.

7. Tensor interaction:

We write the tensor operator S_{12} as

$$S_{12} = 3(\vec{\sigma}_1\hat{r}_{12})(\vec{\sigma}_2\hat{r}_{12}) - \vec{\sigma}_1\vec{\sigma}_2 \qquad 7.1$$

Recoupling this can be written as coupled tensor operators

$$S_{12} = 3\sum_J \hat{J}\Big[\big[\sigma_1\otimes\sigma_2\big]^{(J)}\otimes\big[\hat{r}_{12}\otimes\hat{r}_{12}\big]^{(J)}\Big]^{(0)} + \sqrt{3}\big[\sigma_1\otimes\sigma_2\big]^{(0)} \qquad 7.2$$

with only $J=0$ and $J=2$ contributing. In this, the last term is cancelled against the $J=0$ term in the sum. The remaining term is

$$S_{12} = \sqrt{4\pi}\sqrt{5} < 1010|20 > \Big[Y^2(\hat{r}_{12})\otimes\big[\sigma_1\otimes\sigma_2\big]^{(2)}\Big]^{(0)} \qquad 7.3$$

Separating $\hat{r}_{12}$ and evaluating the Clebsh-Gordon coefficient finally gives

$$\begin{aligned} < (1,\bar{2})_\lambda|V^{R,T}|(3,\bar{4})_\lambda > =& \frac{1}{2\lambda+1}(-)^{(k_1+k_4)}\sum_{\ell_a}\sum_{\ell_b}\frac{2}{\pi}\int q^2dq\ \tilde{V}^T(q) \\ \sqrt{6}\hat{\ell}_a\hat{\ell}_b < \ell_a0\ell_b0|20 > (-)^{(\ell_a+\ell_b)/2}&\begin{Bmatrix} \ell_a & \ell_b & 2 \\ 1 & 1 & \lambda \end{Bmatrix} < 1\|(Y^{\ell_a}\sigma)^{(\lambda)}\|2 >< 4\|(Y^{\ell_b}\sigma)^{(\lambda)}\|3 > \end{aligned} \qquad 7.4$$

where the form factor is

$$\tilde{V}^T(q) = 4\pi\int V^T(r_{12}j_2(qr_{12})r_{12}^2dr_{12} \qquad 7.5$$

8. Term (LS)^2

This term can be brought into a similar form as the previous two. We write

$$V = V^{LS2}(r_{12})\big[\vec{L}\odot\vec{S}\big]\big[\vec{L}\odot\vec{S}\big] = V^{LS2}(r_{12})\sum_j \hat{j}\Big[\big[L^{(1)}\otimes L^{(1)}\big]^{(j)}\otimes\big[S^{(1)}\otimes S^{(1)}\big]^{(j)}\Big]^{(0)} \qquad 8.1$$

Here j can assume the values 0, 1 and 2. While the terms with $j=0$ can immediately be deduced from the previous two, the term with $j=2$ needs to be evaluated new. For $j=0$ we write

$$\big[\vec{L}\otimes\vec{L}\big]^{(0)}\otimes\big[\vec{S}\otimes\vec{S}\big]^{(0)} = \frac{1}{3}\big[\vec{L}\odot\vec{L}\big]\big[\vec{S}\odot\vec{S}\big] = \frac{1}{6}\big[\vec{L}\odot\vec{L}\big](3+\vec{\sigma}_1\cdot\vec{\sigma}_2) \qquad 8.2$$

Thus this part can be added immediately to the previously evaluated interactions by using $V^{L2,eff} = V^{L2}+\frac{1}{2}V^{LS2}$ and $V^{L2\sigma,eff} = V^{L2\sigma}+\frac{1}{6}V^{LS2}$. For $j=1$ we write

$$\sqrt{3}\Big[\big[L^{(1)}\otimes L^{(1)}\big]^{(1)}\otimes\big[S^{(1)}\otimes S^{(1)}\big]^{(1)}\Big]^{(0)} = -\frac{1}{2}\big[\vec{L}\odot\vec{S}\big] \qquad 8.3$$

This contribution is incorporated by substituting $V^{LS,eff} = V^{LS}-\frac{1}{2}V^{LS2}$.

Here it remains to evaluate only the matrix elements of

$$V = V^{LS2}(r_{12})\sqrt{5}\Big[\big[L^{(1)}\otimes L^{(1)}\big]^{(2)}\otimes\big[S^{(1)}\otimes S^{(1)}\big]^{(2)}\Big]^{(0)} \qquad 8.4$$

Using eq.(2.4) we write

$$\big[S^{(1)}\otimes S^{(1)}\big]^{(2)} = \frac{1}{4}\Big(\big[\vec{\sigma}_1\otimes\vec{\sigma}_1\big]^{(2)}+2\big[\vec{\sigma}_1\otimes\vec{\sigma}_2\big]^{(2)}+\big[\vec{\sigma}_2\otimes\vec{\sigma}_2\big]^{(2)}\Big)$$

Noting that one-body matrix elements vanish for the operator $\big[\sigma_1\otimes\sigma_1\big]^{(2)}$ we write the remaining terms as

$$V = V^{LS2}(r_{12})\frac{\sqrt{5}}{2}\Big[\big[L^{(1)}\otimes L^{(1)}\big]^{(2)}\otimes\big[\sigma_1^{(1)}\otimes\sigma_2^{(1)}\big]^{(2)}\Big]^{(0)} \qquad 8.5$$

We assume that similar to 4.4 the operator $\big[L\otimes L\big]^{(2)}$ can be written as a tensor product of variables $\vec{r}_1$ and $\vec{r}_2$ as $\big[U^{(\kappa_1)}(1)\otimes W^{(\kappa_2)}(2)\big]^{(2)}$ Thus we can write

$$\begin{aligned}\Big[\big[\vec{L}\otimes\vec{L}\big]^{(2)}\otimes\big[\vec{\sigma}_1\otimes\vec{\sigma}_2\big]^{(2)}\Big]^{(0)} &= \Big[\big[U^{(\kappa_1)}(1)\otimes W^{(\kappa_2)}(2)\big]\otimes\big[\vec{\sigma}_1\otimes\vec{\sigma}_2\big]^{(2)}\Big]^{(0)}\\ &= \sqrt{5}\sum_k(-)^{(k_2+1)}\begin{Bmatrix}1 & 1 & 2\\ \kappa_1 & \kappa_2 & k\end{Bmatrix}\Big[\big[U^{(\kappa_1)}(1)\otimes\vec{\sigma}_1\big]^{(k)}\odot\big[W^{(\kappa_2)}(2)\otimes\vec{\sigma}_2\big]^{(k)}\Big]^{(0)}\end{aligned} \qquad 8.6$$

We now have to factorize the operator $\big[L\otimes L\big]^{(2)}$ using eq.(1.10) with $j=2$. Again, as they combine with different form factors, we list the three cases separately namely: (a) $J_r=0, J_p=2$, (b) $J_r=2, J_p=0$, and (c) $J_r=2, j_p=2$. Numerically evaluating the 9j-symbol and CG-coefficient we find

$$\begin{aligned}\frac{\sqrt{5}}{2}\big[\vec{L}\otimes\vec{L}\big]^{(2)} = \sqrt{4\pi}\Big(&\frac{\sqrt{5}}{12}r^2\Big[Y^{(0)}(\hat{r})\otimes\big[\vec{\nabla}\otimes\vec{\nabla}\big]^{(2)}\Big]^{(2)}\\ &-\frac{\sqrt{2}}{24}r^2\Big[Y^{(2)}(\hat{r})\otimes\big[\vec{\nabla}\otimes\vec{\nabla}\big]^{(0)}\Big]^{(2)}\\ &+\frac{\sqrt{14}}{12}r^2\Big[Y^{(2)}(\hat{r})\otimes\big[\vec{\nabla}\otimes\vec{\nabla}\big]^{(2)}\Big]^{(2)}\\ &-\frac{1}{4}\sqrt{\frac{5}{3}}r\big[Y^{(1)}(\hat{r})\otimes\vec{\nabla}\big]^{(2)}\Big)\end{aligned} \qquad 5.7$$

In the following we list all the eleven terms that contribute to $\left[L\otimes L\right]^{(2)}$ in their recoupled forms:

$$
\begin{aligned}
Z_1 =&\Big[\big[Y^{\ell_1}(1)\otimes Y^{\ell_2}(2)\big]^{(2)}\otimes\big[\nabla_1\otimes\nabla_1\big]^{(0)}\Big]^{(2)}\\
=&\Big[\big[Y^{\ell_1}\otimes\big[\nabla_1\otimes\nabla_1\big]^{(0)}\big]^{(\ell_1)}\otimes Y^{\ell_2}(2)\Big]^{(2)}\\
Z_2 =&-2\Big[\big[Y^{\ell_1}(1)\otimes Y^{\ell_2}(2)\big]^{(2)}\otimes\big[\nabla_1\otimes\nabla_2\big]^{(0)}\Big]^{(2)}\\
=&\frac{2}{\sqrt{3}}\sum_{L_1,L_2}(-)^{(L_2+\ell_1)}\hat{L}_1\hat{L}_2\begin{Bmatrix} L_1 & L_2 & 2\\ \ell_2 & \ell_1 & 1\end{Bmatrix}\Big[\big[Y^{\ell_1}(1)\nabla_1\big]^{(L_1)}\otimes\big[Y^{\ell_2}(2)\nabla_2\big]^{(L_2)}\Big]^{(2)}\\
Z_3 =&\Big[Y^{\ell_1}(1)\otimes\Big[Y^{\ell_2}(2)\otimes\big[\nabla_2\otimes\nabla_2\big]^{(0)}\Big]^{(\ell_2)}\Big]^{(2)}\\
Z_4 =&\Big[\big[Y^{\ell_1}(1)\otimes Y^{\ell_2}(2)\big]^{(0)}\otimes\big[\nabla_1\otimes\nabla_1\big]^{(2)}\Big]^{(2)}\\
=&\sum_k\delta_{\ell_1,\ell_2}\frac{\hat{k}}{\hat{\ell}_1\sqrt{5}}\Big[\big[Y^{\ell_1}(1)\otimes\big[\nabla_1\otimes\nabla_1\big]^{(2)}\big]^{(k)}\otimes Y^{\ell_2}(2)\Big]^{(2)}\\
Z_5 =&-2\Big[\big[Y^{\ell_1}(1)\otimes Y^{\ell_2}(2)\big]^{(0)}\otimes\big[\nabla_1\otimes\nabla_2\big]^{(2)}\Big]^{(2)}\\
=&2\delta_{\ell_1,\ell_2}\sum_{L_1,L_2}(-)^{(L_1+\ell_1)}\frac{\hat{L}_1\hat{L}_2}{\ell_1}\begin{Bmatrix} 1 & 1 & 2\\ L_2 & L_1 & \ell_1\end{Bmatrix}\Big[\big[Y^{\ell_1}(1)\nabla_1\big]^{(L_1)}\otimes\big[Y^{\ell_2}(2)\nabla_2\big]^{(L_2)}\Big]^{(2)}\\
Z_6 =&\sum_k(-)^{(k+\ell_2)}\delta_{\ell_1,\ell_2}\frac{\hat{k}}{\hat{\ell}_1\sqrt{5}}\Big[Y^{\ell_1}(1)\otimes\big[Y^{\ell_2}(2)\otimes\big[\nabla_2\otimes\nabla_2\big]^{(2)}\big]^{(k)}\Big]^{(2)}\\
Z_7 =&\Big[\big[Y^{\ell_1}(1)\otimes Y^{\ell_2}(2)\big]^{(2)}\otimes\big[\nabla_1\otimes\nabla_1\big]^{(2)}\Big]^{(2)}\\
=&\sqrt{5}\sum_k(-)^{(\ell_1+k)}\hat{k}\begin{Bmatrix} 2 & 2 & 2\\ \ell_2 & \ell_1 & k\end{Bmatrix}\Big[\big[Y^{\ell_1}(1)\otimes\big[\nabla_1\otimes\nabla_1\big]^{(2)}\big]^{(k)}\otimes Y^{\ell_2}(2)\Big]^{(2)}\\
Z_8 =&-2\Big[\big[Y^{\ell_1}(1)\otimes Y^{\ell_2}(2)\big]^{(2)}\otimes\big[\nabla_1\otimes\nabla_2\big]^{(2)}\Big]^{(2)}\\
=&-10\sum_{L_1,L_2}\hat{L}_1\hat{L}_2\begin{Bmatrix} \ell_1 & \ell_2 & 2\\ 1 & 1 & 2\\ L_1 & L_2 & 2\end{Bmatrix}\Big[\big[Y^{\ell_1}(1)\nabla_1\big]^{(L_1)}\otimes\big[Y^{\ell_2}(2)\nabla_2\big]^{(L_2)}\Big]^{(2)}\\
Z_9 =&\sqrt{5}\sum_k\hat{k}\begin{Bmatrix} 2 & 2 & 2\\ \ell_2 & \ell_1 & k\end{Bmatrix}\Big[Y^{\ell_1}(1)\otimes\Big[Y^{\ell_2}(2)\otimes\big[\nabla_2\otimes\nabla_2\big]^{(2)}\Big]^{(k)}\Big]^{(2)}\\
Z_{10} =&\Big[\big[Y^{\ell_1}\otimes Y^{\ell_2}\big]^{(1)}\otimes\nabla_1\Big]^{(2)}\\
=&\sum_k(-)^{(\ell_2+k)}\sqrt{3}\hat{k}\begin{Bmatrix} 1 & 2 & 1\\ \ell_2 & \ell_1 & k\end{Bmatrix}\Big[\big[Y^{\ell_1}(1)\nabla_1\big]^{(k)}\otimes Y^{\ell_2}(2)\Big]^{(2)}\\
Z_{11} =&-\sum_k\sqrt{3}\hat{k}\begin{Bmatrix} 1 & 2 & 1\\ \ell_1 & \ell_2 & k\end{Bmatrix}\Big[Y^{\ell_1}(1)\otimes\big[Y^{\ell_2}(2)\nabla_2\big]^{(k)}\Big]^{(2)}
\end{aligned}
\tag{8.8}
$$

Combining this with with eq.(8.7), we can write the matrix element contributions for the various form factors ($J=0$, $J=1$, and $J=2$) as

$$
\begin{aligned}
<(1\bar{2})_\lambda|V^{R,LS2}_{(J=0)}|(3\bar{4})_\lambda>=&-\frac{\sqrt{5}}{24}\frac{1}{2\lambda+1}(-)^{(k_1+k_4)}\frac{2}{\pi}\int q^2dq\tilde{V}^{LS20}(q)\sum_{\ell,\kappa}\hat{\kappa}\begin{Bmatrix} 1 & 1 & 2\\ \kappa & \ell & \lambda\end{Bmatrix}\\
&\Big(<1\|\big[(Y^\ell\otimes\big[\vec{\nabla}\otimes\vec{\nabla}\big]^{(2)})^{(\kappa)}\vec{\sigma}\big]^{(\lambda)}\|2><4\|(Y^\ell\vec{\sigma})^{(\lambda)}\|3>\\
&+<1\|(Y^\ell\vec{\sigma})^{(\lambda)}\|2><4\|\big[(Y^\ell\otimes\big[\vec{\nabla}\otimes\vec{\nabla}\big]^{(2)})^{(\kappa)}\vec{\sigma}\big]^{(\lambda)}\|3>\Big)
\end{aligned}
\tag{8.9}
$$

$$< (1\bar{2})_\lambda|V^{R,LS2}_{(J=0)Z=2}|(3\bar{4})_\lambda >= -\frac{5}{12}\frac{1}{2\lambda+1}(-)^{(k_1+k_4)}\sum_{\ell,\ell_a,\ell_b}(-)^{(\ell_a+\ell_b)}\hat{\ell}_a\hat{\ell}_b\begin{Bmatrix} 1 & 1 & 2 \\ \ell_a & \ell_b & \ell \end{Bmatrix}\begin{Bmatrix} 1 & 1 & 2 \\ \ell_a & \ell_b & \lambda \end{Bmatrix}$$

$$\frac{2}{\pi}\int q^2 dq \tilde{V}^{LS20}(q) \times < 1\|\big[(Y^\ell\vec{\nabla})^{(\ell_a)}\vec{\sigma}\big]^{(\lambda)}\|2 >< 4\|\big[(Y^\ell\vec{\nabla})^{(\ell_b)}\vec{\sigma}\big]^{(\lambda)}\|3 >\Big) \qquad 8.10$$

The contributions from terms Z_{10}, and Z_{11}, the $J = 1$ terms are

$$< (1\bar{2})_\lambda|V^{R,LS2}_{(J=1)}|(3\bar{4})_\lambda >= \frac{5}{\sqrt{12}}\frac{1}{2\lambda+1}(-)^{(k_1+k_4)}\sum_{\ell_a,\ell_b,\kappa}(-)^\kappa\hat{\kappa} < \ell_a 0\ell_b 0|10 > \begin{Bmatrix} 1 & 1 & 2 \\ \ell_b & \kappa & \ell_a \end{Bmatrix}\begin{Bmatrix} 1 & 1 & 2 \\ \ell_b & \kappa & \lambda \end{Bmatrix}$$

$$\frac{2}{\pi}\int q^2 dq \tilde{V}^{LS2,1}(q)\hat{\ell}_a\hat{\ell}_b(-)^{(\ell_a-\ell_b-1)/2}\Big(< 1\|\big[(Y^{\ell_a}\vec{\nabla})^{(\kappa)}\vec{\sigma}\big]^{(\lambda)}\|2 >< 4\|(Y^{\ell_b}\vec{\sigma})^{(\lambda)}\|3 >$$

$$+ < 1\|(Y^{\ell_b}\vec{\sigma})^{(\lambda)}\|2 >< 4\|\big[(Y^{\ell_a}\vec{\nabla})^{(\kappa)}\vec{\sigma}\big]^{(\lambda)}\|3 >\Big) \qquad 8.11$$

Finally, the $J = 2$ terms are from Z_1, Z_2, Z_3, Z_7, Z_8, and Z_9.

$$< (1\bar{2})_\lambda|V^{R,LS2}_{(J=2)}|(3\bar{4})_\lambda >= \frac{1}{2\lambda+1}(-)^{(k_1+k_4)}\frac{2}{\pi}\int q^2 dq \tilde{V}^{LS2,2}(q)\sum_{\ell_1,\ell_2}\hat{\ell}_1\hat{\ell}_2 < \ell_1 0\ell_2 0|20 > (-)^{(\ell_1+\ell_2)/2}$$

$$\Bigg\{\sum_\kappa \frac{\sqrt{70}}{24}\hat{\kappa}\begin{Bmatrix} 1 & 1 & 2 \\ \kappa & \ell_1 & \lambda \end{Bmatrix}\begin{Bmatrix} 2 & 2 & 2 \\ \ell_1 & \ell_2 & \kappa \end{Bmatrix}(-)^{(\kappa+\ell_1)}$$

$$\Big(< 1\|\Big[\big[Y^{\ell_2}\otimes\big[\vec{\nabla}\otimes\vec{\nabla}\big]^{(2)}\big]^{(\kappa)}\vec{\sigma}\Big]^{(\lambda)}\|2 >< 4\|(Y^{\ell_1}\vec{\sigma})^{(\lambda)}\|3 >$$

$$+ < 1\|(Y^{\ell_1}\vec{\sigma})^{(\lambda)}\|2 >< 4\|\Big[\big[Y^{\ell_2}\otimes\big[\vec{\nabla}\otimes\vec{\nabla}\big]^{(2)}\big]^{(\kappa)}\vec{\sigma}\Big]^{(\lambda)}\|3 >\Big) \qquad 8.12$$

$$-\frac{\sqrt{2}}{24}\begin{Bmatrix} 1 & 1 & 2 \\ \ell_1 & \ell_2 & \lambda \end{Bmatrix}\Big(< 1\|\Big[\big[Y^{\ell_2}\otimes\big[\vec{\nabla}\otimes\vec{\nabla}\big]^{(0)}\vec{\sigma}\Big]^{(\lambda)}\|2 >< 4\|(Y^{\ell_1}\vec{\sigma})^{(\lambda)}\|3 >$$

$$+ < 1\|(Y^{\ell_1}\vec{\sigma})^{(\lambda)}\|2 >< 4\|\Big[\big[Y^{\ell_2}\otimes\big[\vec{\nabla}\otimes\vec{\nabla}\big]^{(0)}\vec{\sigma}\Big]^{(\lambda)}\|3 >\Big)\Bigg\}$$

$$< (1\bar{2})_\lambda|V^{R,LS2}_{(J=2)}|(3\bar{4})_\lambda >= -\frac{1}{2\lambda+1}(-)^{(k_1+k_4)}\frac{2}{\pi}\int q^2 dq \tilde{V}^{LS2,2}(q)\sum_{\ell_1,\ell_2}\hat{\ell}_1\hat{\ell}_2 < \ell_1 0\ell_2 0|20 > (-)^{(\ell_1+\ell_2)/2}$$

$$\sum_{\ell_a,\ell_b}\hat{\ell}_a\hat{\ell}_b\begin{Bmatrix} 1 & 1 & 2 \\ \ell_a & \ell_b & \lambda \end{Bmatrix}\Bigg(\sqrt{\frac{2}{3}}\frac{1}{6}\begin{Bmatrix} \ell_2 & \ell_1 & 2 \\ \ell_a & \ell_b & \lambda \end{Bmatrix} + \sqrt{14}\frac{5}{6}(-)^{(\ell_b+\ell_2)}\begin{Bmatrix} \ell_1 & \ell_2 & 2 \\ 1 & 1 & 2 \\ \ell_a & \ell_b & 2 \end{Bmatrix}\Bigg)$$

$$< 1\|\big[(Y^{\ell_1}\vec{\nabla})^{(\ell_a)}\vec{\sigma}\big]^{(\lambda)}\|2 >< 4\|\big[(Y^{\ell_2}\vec{\nabla})^{(\ell_b)}\vec{\sigma}\big]^{(\lambda)}\|3 > \qquad 8.13$$

9. The iso-spin component

The iso-spin dependence of the matrix elements is best worked out using the (pp) coupled matrix elements. In that case all we have to do is to work out the expectation value of the iso-spin operator. We do this here for the four iso-spin operators that appear in the V-18 potential of Wiringa et al. and the four possible matrix elements ($\mathbf{1}, \vec{\tau}_1\vec{\tau}_2, T_{12} = 3\tau_{z1}\tau_{z2} - \vec{\tau}_1\vec{\tau}_2$, and $\tau_{z1} + \tau_{z2}$):

$$
\begin{array}{ccc}
<(pp)|\mathbf{1}|(pp)>=1 & , & <(pp)|\tau_{z1}+\tau_{z2}|(pp)>=+2 \\
<(pn)|\mathbf{1}|(pn)>=1 & , & <(pn)|\tau_{z1}+\tau_{z2}|(pn)>=0 \\
<(pn)|\mathbf{1}|(np)>=0 & , & <(pn)|\tau_{z1}+\tau_{z2}|(np)>=0 \\
<(nn)|\mathbf{1}|(nn)>=1 & , & <(nn)|\tau_{z1}+\tau_{z2}|(nn)>=-2
\end{array}
$$

$$
\begin{array}{ccc}
<(pp)|\vec{\tau}_1\vec{\tau}_2|(pp)>=1 & , & <(pp)|T_{12}|(pp)>=2 \\
<(pn)|\vec{\tau}_1\vec{\tau}_2|(pn)>=-1 & , & <(pn)|T_{12}|(pn)>=-2 \\
<(pn)|\vec{\tau}_1\vec{\tau}_2|(np)>=2 & , & <(pn)|T_{12}|(np)>=-2 \\
<(nn)|\vec{\tau}_1\vec{\tau}_2|(nn)>=1 & , & <(nn)|T_{12}|(nn)>=2
\end{array}
$$

Thus we calculate the matrix elements
for $<(pp)|V|(pp)>$

$$<(12)|V|(34)>=<(12)|V+V^{\tau\tau}+2V^{tT}+2V^{\tau z}+V^{Coul}|(34)>$$

for $<(pn)|V|(pn)>$

$$<(12)|V|(34)>=<(12)|V-V^{\tau\tau}-2V^{tT}|(34)>$$

for $<(pn)|V|(np)>$

$$<(12)|V|(34)>=<(12)|2V^{\tau\tau}-2V^{tT}|(34)>$$

and for $<(nn)|V|(nn)>$

$$<(12)|V|(34)>=<(12)|V+V^{\tau\tau}+2V^{tT}-2V^{\tau z}|(34)>$$

10.CM-correction terms

From the center of mass corrections we have two additional terms that must be included in the two-body matrix elements of the interaction. These are:

$$-\frac{\vec{p}_1\cdot\vec{p}_2}{mA} = +\frac{\hbar\vec{\nabla}_1\cdot\vec{\nabla}_2}{mA}$$

and

$$\frac{\vec{r}_1\cdot\vec{r}_2}{mA}$$

These can be computed using (2.1). These matrix elements contribute only for $\lambda = 1$ and can be computed with eqn (3.3) resulting in

$$-<1\bar{2}|\vec{p}_1\vec{p}_2|3\bar{4}>=\frac{\hbar^2}{2mA}(-)^{k_1}\frac{1}{\sqrt{3}}<1\|\nabla\|2>(-)^{k_4}\frac{1}{\sqrt{3}}<4\|\nabla\|3>$$

with

$$(-)^{k_1}\frac{1}{\sqrt{3}}<1\|\vec{\nabla}\|2>=\hat{j}_1\hat{j}_2<j_1\frac{1}{2}j_2\frac{-1}{2}|10><R_1(r)|\frac{d}{dr}+\frac{1}{2}(\ell_2-\ell_1)(3\ell_2-\ell_1+1)\frac{1}{r}|R_2(r)>$$

and

$$(-)^{k_1}\frac{1}{\sqrt{3}}<1\|\vec{r}\|2>=\frac{-1}{3}\hat{j}_1\hat{j}_2<j_1\frac{1}{2}j_2\frac{-1}{2}|10><R_1(r)|r|R_2(r)>$$

11. The radial integrals

We assume that each wave function is expanded in harmonic oscillator functions corresponding to the oscillator length parameter ok.

$$R_{i,l}(r) = \sum_n A_n^i H_{n,l}(r) \qquad 10.1$$

As such each of the radial wave functions can be written as a polynomial multiplied by a gaussian of argument $-(r/ok\sqrt{2})^2$. We now introduce the variable $x = \sqrt{2}r/ok$. With this, each Radial wave function can be written in terms of x as

$$R_{i,l}(x) = \bar{P}_{i,l}(x)e^{-x^2/4} \qquad 10.2$$

where $\bar{P}_{i,l}$ is a ploynomial in x. In terms of x the wave functions are normalized such that

$$\int R_{i,l}^2(x)x^2dx = (\frac{\sqrt{2}}{ok})^3 \qquad 10.3$$

In order to carry out the Fourier transform we use the fact that the Fourier transforms of the harmonic oscillator functions are again harmonic oscillator functions in k-space. Thus we expand the product of two radial functions as

$$R_{i,l_i}(x)R_{j,l_j}(x) = \sum_n B_n^{i,j} H_{n,L}(x) \qquad 10.4$$

Using Gaussian integration, the expansion coefficients can be written as

$$B_n^{i,j} = \sum_k L_{n,L}(x_k)\bar{P}_{i,l_i}(x_k)\bar{P}_{j,l_j}(x_k)x_k^2 w_k \qquad 10.5$$

with this we write the integral

$$\int R_{i,l_i}(r)j_L(qr)R_{j,l_j}(r)r^2dr = \sum_n B_n^{i,j} H_{n,L}(\bar{q}) \qquad 10.6$$

where $\bar{q} = \sqrt{2}ok\ q$. The radial integrals can now be computed according to

$$\begin{aligned}
&\frac{2}{\pi}\int q^2dq\ V(q)\langle R_{i,l_i}|j_L(qr)|R_{j,l_j}\rangle\langle R_{s,l_s}|j_K(qr)|R_{t,l_t}\rangle = \\
&= \sum_n\sum_m B_n^{i,j}B_m^{s,t}\frac{2}{\pi}\int q^2dq\ V(q)H_{n,L}(\bar{q})H_{m,K}(\bar{q}) \qquad 10.7\\
&=: \sum_n\sum_m B_n^{i,j}B_m^{s,t}I_{n,m}^{L,K}
\end{aligned}$$

Inserting the expression for the expansion coefficients B we find

$$\begin{aligned}
&\frac{2}{\pi}\int q^2dq\ V(q)\langle R_{i,l_i}|j_L(qr)|R_{j,l_j}\rangle\langle R_{s,l_s}|j_K(qr)|R_{t,l_t}\rangle = \\
&= \frac{2}{\pi}\sum_k\sum_q \bar{P}_{i,l_i}(x_k)\bar{P}_{j,l_j}(x_k)x_k^2w_k\bar{P}_{s,l_s}(x_q)\bar{P}_{t,l_t}(x_q)x_q^2w_q\sum_{m,n}I_{n,m}^{L,K}H_{n,L}(x_k)H_{m,K}(x_q) \qquad 10.8\\
&=: \sum_k\sum_q \bar{P}_{i,l_i}(x_k)\bar{P}_{j,l_j}(x_k)\bar{P}_{s,l_s}(x_q)\bar{P}_{t,l_t}(x_q)\ S_{k,q}^{L,K}
\end{aligned}$$

With this notation all terms can be computed as

$$\begin{aligned}
< (1\bar{2})_\lambda|V^{tot}|(3\bar{4})_\lambda >= &\sum_{i,j}\bar{P}_1(x_i)\bar{P}_2(x_i)S_{i,j}^1\bar{P}_3(x_j)\bar{P}_4(x_j)\\
&+\bar{P}_1(x_i)\bar{P}_2'(x_i)S_{i,j}^2\bar{P}_3(x_j)\bar{P}_4(x_j)\\
&+\bar{P}_1(x_i)\bar{P}_2(x_i)S_{i,j}^3\bar{P}_3'(x_j)\bar{P}_4(x_j) \qquad 10.9\\
&+\bar{P}_1(x_i)\bar{P}_2''(x_i)S_{i,j}^4\bar{P}_3(x_j)\bar{P}_4(x_j)\\
&+\bar{P}_1(x_i)\bar{P}_2(x_i)S_{i,j}^5\bar{P}_3''(x_j)\bar{P}_4(x_j)\\
&+\bar{P}_1(x_i)\bar{P}_2'(x_i)S_{i,j}^6\bar{P}_3'(x_j)\bar{P}_4(x_j)
\end{aligned}$$

correspondingly, the exchange contribution can be written as

$$\begin{aligned} < (1\bar{2})_\lambda|V^{tot}|(3\bar{4})_\lambda >= \sum_{i,j} & \bar{P}_1(x_i)\bar{P}_3(x_i)E^1_{i,j}\bar{P}_2(x_j)\bar{P}_4(x_j) \\ & +\bar{P}_1(x_i)\bar{P}'_3(x_i)E^2_{i,j}\bar{P}_2(x_j)\bar{P}_4(x_j) \\ & +\bar{P}_1(x_i)\bar{P}_3(x_i)E^3_{i,j}\bar{P}'_2(x_j)\bar{P}_4(x_j) \\ & +\bar{P}_1(x_i)\bar{P}''_3(x_i)E^4_{i,j}\bar{P}_2(x_j)\bar{P}_4(x_j) \\ & +\bar{P}_1(x_i)\bar{P}_3(x_i)E^5_{i,j}\bar{P}''_2(x_j)\bar{P}_4(x_j) \\ & +\bar{P}_1(x_i)\bar{P}'_3(x_i)E^6_{i,j}\bar{P}'_2(x_j)\bar{P}_4(x_j) \end{aligned} \quad 10.10$$

As the marices S, and E only depend on the angular momenta involved but not on the quantum numbers n_1,n_2,n_3, and n_4, the strategy is to compute the matrices first and then compute all the matrix elements with the same angular momentum combinations. This way a large number of matrix elements can be computed in which orbits differ only by the quantum number n.
This strategy is essential in computing the G-matrix interaction in calculating the term:

$$\sum_K \sum_{p_3,p_4} Z^K_{p_3p_4,h_1h_2} < (p_3p_4)_K|V|(p_1p_2)_K >$$

[1] R.B.Wiringa, V.G.J.Stoks, and R.Schiavilla, *Phys. Rev.* **C 51** (1995), 38
[2] S.Krewald, V.Klemt, J.Speth, and A.Faessler, 'On the Use of Skyrme-Forces in Selfconsistent RPA-Calculations' Jülich, Germany