Center-of-mass corrections reexamined: a many-body expansion approach

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(August 4, 2006)

A many-body expansion for the computation of the charge form factor in the center-of-mass system is proposed. For convergence testing purposes, we apply our formalism to the case of the harmonic oscillator shell model, where an exact solution exists. We also work out the details of the calculation involving realistic nuclear wave functions. Results obtained for the Argonne $v18$ two-nucleon and Urbana-IX three-nucleon interactions are reported. No corrections due to the meson-exchange charge density are taken into account.

PACS numbers: 21.10.Ft, 21.60.-n, 21.60.Gx, 21.60.Cs, 21.60.Jz

I. INTRODUCTION

One of the successes of the shell-model picture has been the ability to calculate self-consistent densities for nuclear ground states that not only reproduce experimental binding energies but also experimental charge radii of these nuclei and generally nuclear charge densities. The excellent agreement or remaining discrepancies have been a cornerstone for advancing our understanding of the nuclear wave function. In particular, the ability to predict both heavy and light nuclei is taken as a confirmation of the quality of the effective nuclear interaction used in the calculations. For that reason it is useful to examine the accuracy with which the nuclear densities can be calculated.

For the proper description of the scattering process, one assumes a nuclear wave function that factorizes into a nuclear center-of-mass wave function, which is taken to be a plane wave, and an intrinsic wave function of coordinates relative to the center-of-mass. The difficulty lies in the ansatz of the wave function as a Slater determinant. Such a wave function generally does not factorize into a center-of-mass wave function and a wave function for the nucleus relative to its center-of-mass. Furthermore, for the cases where it factorizes, the center-of-mass wave function is not a plane wave. While this is negligible for heavy nuclei, it is a significant correction for nuclei like ^{16}O.

This problem has been known for a long time. It can be solved exactly for a single Slater determinant of harmonic oscillator single-particle wave functions. In that case it has been shown that the wave function factorizes with a center-of-mass wave function being a Gaussian. This allows us to calculate the form factor, i.e. the Fourier transform of the density, in the form

$$F_{sd}(q) \;=\; e^{-\frac{1}{4}b^2q^2/A}\; F_{int}(q) \tag{1.1}$$

given in terms of the harmonic oscillator length parameter b. The calculation usually gives the form factor of the one-body density labeled $F_{sd}(q)$, whereas the experiment requires the form factor with respect to the center-of-mass, labeled $F_{int}(q)$. Because of this exact result, it has been customary to apply such a correction also in cases where the single-particle wave functions are not harmonic oscillator wave functions and where the presence of correlations has been substituted by an effective interaction.

An alternate way to deal with this is to calculate directly the form factor in the center-of-mass system. This way the operator can be written as a series of one-body, two-body, ..., to A-body terms. In this paper we first compare such an expansion with the exact result for the case where such a result is available. We then apply the same expansion to a realistic wave function of ^{16}O [**?**] and compare it to the corrections implied by Eq. (**??**). This nuclear wave function was derived for ^{16}O using correlations of the form $exp(\mathbf{S})$ together with the Argonne $v18$ potential [**?**] that provides an excellent fit to the nucleon-nucleon scattering and thus must be considered as a realistic interaction. The nuclear interaction also includes a phenomenological (Urbana-IX) three-nucleon interaction [**?**]. Thus, in this paper we hope to shed some light on the reliability of such center-of-mass corrections.

II. THE FORM FACTOR OF THE DENSITY

The charge form factor at momentum transfer $\vec{q}$ is given in Born approximation [**?**] by

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$$F_{int}(\vec{q}) \;=\; \langle \phi_0 \,|\, \sum_k f_k(q^2)\; e^{i\vec{q}\cdot\vec{r'}_k} \,|\, \phi_0 \rangle \;,$$

where ϕ_0 is the translationally invariant ground state, $\vec{r'}_k$ the distance from the center-of-mass to the kth "point" nucleon, and $f_k(q^2)$ the nucleon form factor, which takes into account the finite size of the nucleon k.

The center-of-mass correction has to do with the fact that the origin of the shell model is not the same as the center-of-mass of the nucleus. Since the many-body Hamiltonian is not translationally invariant, then the model ground state $\Phi_0^{(M)}$ is not translationally invariant either, and thus can lead to an incorrect description of observables, especially in small A nuclei.

What we need to establish is the relationship between the model quantities expressed in terms of the coordinates of the laboratory system ($\vec{r}_k$, $k = 1 \ldots A$) and the intrinsic ones ($\vec{r'}_k = \vec{r}_k - \vec{R}_{cm}$, $k = 1 \ldots A-1$), measured from the center-of-mass of the nucleus

$$\vec{R}_{cm} \;=\; \frac{1}{A} \sum_{k=1}^{A} \vec{r}_k \;.$$

Formally, this may be viewed as a change of coordinates, from the coordinates of the laboratory system $\vec{r}_k$ to the coordinates of the center-of-mass system $\{\vec{R}_{cm},\, \vec{r'}_k\}$, followed by the removal of the dependence upon $\vec{R}_{cm}$ from the model wave function $\Phi_0^{(M)}$, i.e. we have to construct the intrinsic wave function [**?**]

$$\phi_0^{(M)}(\vec{r'}_k) \;=\; \int G(\vec{R}_{cm})\, \Phi_0^{(M)}(\vec{R}_{cm}, \vec{r'}_k)\, d\vec{R}_{cm} \;,$$

independent of $\vec{R}_{cm}$, for an arbitrary function $G(\vec{R}_{cm})$. Note here that, in this formalism, the Gartenhaus-Schwartz transformation [**?**,**?**] corresponds to taking $G(\vec{R}_{cm}) = \delta(\vec{R}_{cm})$. Since there is no reason to choose a particular $G(\vec{R}_{cm})$, it has been pointed out that the center-of-mass correction for a given model wave function is not uniquely defined [**?**]. Nevertheless, the various recipes yield the same result in the limit of the exact wave function of a free nucleus [**?**].

The exact nuclear wave function Φ_0 consists of two factors, one of which is a plane wave in the center-of-mass coordinate, $e^{i\vec{P}\cdot\vec{R}_{cm}}$, the other being the intrinsic wave function ϕ_0 of the relative coordinates [**?**] $\vec{r'}_k$,

$$\Phi_0(\vec{r}_1 \cdots \vec{r}_A) \;=\; e^{i\vec{P}\cdot\vec{R}_{cm}}\; \phi_0(\vec{r'}_1 \cdots \vec{r'}_{A-1}) \;.$$

For an approximate model wave function $\Phi_0^{(M)}$, however, all we can hope for is to be able to obtain the decomposition

$$\Phi_0^{(M)}(\vec{r}_1 \cdots \vec{r}_A) \;=\; \phi_{cm}(\vec{R}_{cm})\; \phi_0^{(M)}(\vec{r'}_1 \cdots \vec{r'}_{A-1}) \;,$$

which is approximately correct to the extent that the motion of the intrinsic coordinates and the center-of-mass are not correlated. Only then, the factorization

$$F_{sd}(\vec{q}) \;=\; F_{cm}(\vec{q})\; F_{int}(\vec{q}) \qquad (2.1)$$

is possible. To that approximation, and assuming that the model provides indeed a good description of the internal structure of the nucleus ($\Phi_0 = \Phi_0^{(M)}$ [**?**]), Eq. (**??**) is valid with [**?**]

$$F_{int}(\vec{q}) \;=\; \langle \Phi_0^{(M)} \,|\, \sum_k f_k(q^2)\; e^{i\vec{q}\cdot(\vec{r}_k - \vec{R}_{cm})} \,|\, \Phi_0^{(M)} \rangle \qquad (2.2)$$

and

$$F_{cm}(\vec{q}) \;=\; \langle \Phi_0^{(M)} \,|\, e^{i\vec{q}\cdot\vec{R}_{cm}} \,|\, \Phi_0^{(M)} \rangle \;.$$

The form factor (**??**) can now be calculated directly by carrying out an expansion in terms of many-body operators

$$F_{int}(\vec{q}) = \sum_k\; f_k(q^2)\; \left\langle\; e^{i\vec{q}\cdot\vec{r}_k (A-1)/A} \prod_{m\neq k} e^{-i\vec{q}\cdot\vec{r}_m/A} \right\rangle \;. \qquad (2.3)$$

Each exponential in Eq. (**??**) can be expressed in terms of the one-body operator which we define by

$$f(\vec{q}\cdot\vec{r}_m) = e^{i\vec{q}\cdot\vec{r}_m} - 1 \;.$$

With this we write the form factor as

$$\begin{aligned} F_{int}(\vec{q}) \;&=\; \sum_k\; f_k(q^2) \\ &\times \left\langle e^{i\vec{q}\cdot\vec{r}_k (A-1)/A} \prod_{m\neq k} (1 + f^*(\vec{q}\cdot\vec{r}_m/A)) \right\rangle \end{aligned}$$

or

$$\begin{aligned} F_{int}(\vec{q}) \;&=\; \sum_k\; f_k(q^2)\; \left\langle e^{i\vec{q}\cdot\vec{r}_k (A-1)/A} \right\rangle \\ &+ \sum_k\; f_k(q^2) \sum_{m\neq k} \left\langle e^{i\vec{q}\cdot\vec{r}_k (A-1)/A}\; f^*(\vec{q}\cdot\vec{r}_m/A) \right\rangle \\ &+ \frac{1}{2} \sum_k\; f_k(q^2) \sum_{m,n\neq k} \\ &\times \left\langle e^{i\vec{q}\cdot\vec{r}_k (A-1)/A} f^*(\vec{q}\cdot\vec{r}_m/A) f^*(\vec{q}\cdot\vec{r}_n/A) \right\rangle + \cdots \;. \end{aligned} \qquad (2.4)$$

We intend to apply our formalism to the particular case of doubly magic nuclei (^{16}O). Thus, we can use the spherical symmetry of the nucleus to simplify calculations, in the sense that the form factor $F_{int}(\vec{q})$ should be spherically symmetric too, and we can in turn average the form factor over the directions of $\vec{q}$. Then, we introduce

$$F_{int}^{(av)}(q) = \frac{1}{4\pi} \int F_{int}(\vec{q})\; d\Omega_q \;.$$

This allows us to write the different terms in equation (**??**) using the second quantization formalism, as follows:

1. One-body term

$$\sum_{\alpha\beta} f_\alpha(q^2) \ \langle\alpha \,|\, \mathcal{O}(q,\vec{r}_1)|\,\beta\rangle \ \mathbf{a}^\dagger_\alpha \mathbf{a}_\beta \,, \qquad (2.5)$$

with

$$\mathcal{O}(q,\vec{r}_1) \ = \ j_0(qr_1(A-1)/A) \ .$$

2. Two-body term

$$\sum_L (2L+1) \times \sum_{\alpha\beta\gamma\delta} f_\alpha(q^2) \ \langle\alpha\beta \,|\, \mathcal{O}(q,\vec{r}_1,\vec{r}_2) \,|\, \gamma\delta\rangle \ \mathbf{a}^\dagger_\alpha \mathbf{a}^\dagger_\beta \mathbf{a}_\delta \mathbf{a}_\gamma \,, \qquad (2.6)$$

with

$$\mathcal{O}(q,\vec{r}_1,\vec{r}_2) = j_L(qr_1(A-1)/A)\, f_L(qr_2/A) \times \left(C^{(L)}_1 \odot C^{(L)}_2\right) \ .$$

3. Three-body term

$$\sum_{L_1L_2L_3} i^{L_1-L_2-L_3}(2L_2+1)(2L_3+1)\langle L_3 0 \, L_2 0 \,|\, L_1 0\rangle \times \sum_{\alpha\beta\gamma\delta\theta\zeta} f_\alpha(q^2) \ \langle\alpha\beta\gamma \,|\, \mathcal{O}(q,\vec{r}_1,\vec{r}_2,\vec{r}_3) \,|\, \delta\theta\zeta\rangle \times \mathbf{a}^\dagger_\alpha \mathbf{a}^\dagger_\beta \mathbf{a}^\dagger_\gamma \mathbf{a}_\zeta \mathbf{a}_\theta \mathbf{a}_\delta \,, \qquad (2.7)$$

with

$$\mathcal{O}(q,\vec{r}_1,\vec{r}_2,\vec{r}_3) = j_{L_1}(qr_1(A-1)/A)\, f_{L_2}(qr_2/A)\, f_{L_3}(qr_3/A) \times \left(C^{(L_1)}_1 \odot \left[C^{(L_2)}_2 \otimes C^{(L_3)}_3\right]^{(L_1)}\right) \ ,$$

where we have introduced $f_l(qr) = j_l(qr) - \delta_{l0}$. Here $j_l(qr)$ and $C^{(l)}_m = \sqrt{\frac{4\pi}{2l+1}}\, Y_{lm}(\hat{r})$ are the spherical Bessel functions of order l and the unnormalized spherical harmonics of rank l and component m, respectively. Greek letters label the single-particle states $|\alpha\rangle = |n_\alpha\,(l_\alpha s_\alpha) j_\alpha m_{j_\alpha}; \tau_\alpha m_{\tau_\alpha}\rangle$, with $s = \frac{1}{2}$, $\tau = \frac{1}{2}$, $j = l \pm \frac{1}{2}$ and $m_\tau = +\frac{1}{2}(-\frac{1}{2})$ – for a proton (neutron). As a final remark, note that the conversion to second quantization allows for all restrictions in the sums (**??**) to be dropped.

III. HARMONIC OSCILLATOR SHELL-MODEL CALCULATION

Equation (**??**) is always exact if $\Phi^{(M)}_0$ is expressed in terms of harmonic oscillator wave functions, provided that the center-of-mass wave function ϕ_{cm} is in one given harmonic oscillator state. Then, the extraction of the center-of-mass coordinate can be done analytically. Elliott and Skyrme [**?**] have shown that if the shell-model states are non-spurious, then the center-of-mass moves in its ground state and is described by the $1s$ harmonic oscillator wave function

$$\phi_{cm}(\vec{R}_{cm}) \ = \ \left(\frac{A^3}{\pi^3\, b^6}\right)^{\frac{1}{4}} \ \exp\left[-\frac{A\, R^2_{cm}}{2\, b^2}\right] \ ,$$

where b is the harmonic oscillator length parameter. The center-of-mass form factor can also be evaluated explicitly

$$F_{cm}(\vec{q}) \ = \ e^{-\frac{1}{4}b^2q^2/A} \ .$$

The correct translation-invariant form factor is thus given in terms of the shell-model form factor by

$$F_{int}(\vec{q}) \ = \ e^{\frac{1}{4}b^2q^2/A} \ F_{sd}(\vec{q}) \,, \qquad (3.1)$$

i.e. F_{sd} must be corrected by dividing by $F_{cm}(q)$. Note that since the uniqueness of the procedure of carrying out the center-of-mass corrections has been questioned, the use of Eq. (**??**) has been suggested even in the case of a more general nuclear structure model [**?**].

We exploit the analytical nature of these results by testing how fast the many-body expansion (**??**) converges. The shell-model wave function $\Phi^{(M)}_0$ for the harmonic oscillator potential is an independent particle wave function, represented by a simple Slater determinant of single-particle orbits. This state is what we call the uncorrelated ground state $|0\rangle$. By taking the expectation value in the model ground state $\Phi^{(M)}_0 = |0\rangle$ of the one-, two- and three-body operators in Eqs. (**??**), (**??**), and (**??**), the following relevant expectation values are obtained

$$\begin{aligned}
\langle 0 \,|\, \mathbf{a}^\dagger_\alpha \mathbf{a}_\beta \,|\, 0\rangle &= \delta_{\alpha\beta} \\
\langle 0 \,|\, \mathbf{a}^\dagger_\alpha \mathbf{a}^\dagger_\beta \mathbf{a}_\delta \mathbf{a}_\gamma \,|\, 0\rangle &= \delta_{\alpha\gamma}\delta_{\beta\delta} \ - \ \delta_{\alpha\delta}\delta_{\beta\gamma} \\
\langle 0 \,|\, \mathbf{a}^\dagger_\alpha \mathbf{a}^\dagger_\beta \mathbf{a}^\dagger_\gamma \mathbf{a}_\zeta \mathbf{a}_\theta \mathbf{a}_\delta \,|\, 0\rangle &= \delta_{\alpha\delta}\,(\delta_{\beta\theta}\delta_{\gamma\zeta} \ - \ \delta_{\beta\zeta}\delta_{\gamma\theta}) \\
&\quad - \delta_{\alpha\theta}\,(\delta_{\beta\delta}\delta_{\gamma\zeta} \ - \ \delta_{\beta\zeta}\delta_{\gamma\delta}) \\
&\quad + \delta_{\alpha\zeta}\,(\delta_{\beta\delta}\delta_{\gamma\theta} \ - \ \delta_{\beta\theta}\delta_{\gamma\delta}) \ .
\end{aligned}$$

Using these results and following a straightforward but laborious calculation, the translation-invariant form factor for the harmonic oscillator shell model can be computed completely up to the third-order in the many-body expansion (**??**). The various components involved are presented here by their corresponding term of origin in the many-body expansion. Summations over all (nlj) indices are implicit. Notations are discussed in an Appendix.

a. One-body term. There is only one contribution to the one-body term of $F_{int}^{(av)}(q)$

$$\mathrm{HO1} = f_{nlj}(q^2)\, I^{(1)\,0}_{nl,nl}\,.$$

Note that in the previous equation, HO1 is actually the Fourier transform of the one-body density folded with the appropriate nucleon form factor, i.e.

$$\mathrm{HO1} = f_p(q^2)\ \int_0^\infty \rho_0^{(p)}(r)\ j_0(\tfrac{A-1}{A}qr)\ r^2\,dr + f_n(q^2)\ \int_0^\infty \rho_0^{(n)}(r)\ j_0(\tfrac{A-1}{A}qr)\ r^2\,dr\,,$$

where $\rho_0^{(p)}(r)$ and $\rho_0^{(n)}(r)$ are the proton and neutron one-body densities, respectively, corresponding to the uncorrelated ground state $|0\rangle$.

b. Two-body term. Two components contribute to the two-body term of $F_{int}^{(av)}(q)$

* one component corresponding to the direct contraction $\delta_{\alpha\gamma}\delta_{\beta\delta}$

$$\mathrm{HO2_{dr}} = f_{n_1l_1j_1}(q^2) I^{(1)\,0}_{n_1l_1j_1,n_1l_1j_1} I^{(2)\,0}_{n_2l_2j_2,n_2l_2j_2}\,;$$

* one component associated with the exchange contraction $\delta_{\alpha\gamma}\delta_{\beta\delta}$

$$\mathrm{HO2_{ex}} = f_{n_1l_1j_1,\,n_2l_2j_2}(q^2)\ \sum_L (2L+1) \times \bar{I}^{(1)\,L}_{n_1l_1j_1,n_2l_2j_2}\,\bar{I}^{(2)\,L}_{n_1l_1j_1,n_2l_2j_2}\,;$$

where the pair of indices of the nucleon form factor $f(q^2)$ indicate that the two orbits denoted as $(n_1l_1j_1)$ and $(n_2l_2j_2)$ have the same isospin.

c. Three-body term. The three-body term contains six contributions to $F_{int}^{(av)}$, out of which two are identical due to the fact that, in Eq. (**??**), the radial and angular parts of the operator dependent upon the coordinates of the second nucleon are the same as the radial and angular parts of the operator dependent upon the coordinates of the third nucleon. The different components of the three-body term (**??**) are listed below

* *term 3.1* $(\delta_{\alpha\delta}\delta_{\beta\theta}\delta_{\gamma\zeta})$

$$\mathrm{HO3_1} = f_{n_1l_1j_1}(q^2) I^{(1)\,0}_{n_1l_1j_1,n_1l_1j_1} I^{(2)\,0}_{n_2l_2j_2,n_2l_2j_2} I^{(2)\,0}_{n_3l_3j_3,n_3l_3j_3}\,;$$

* *term 3.2* $(\delta_{\alpha\delta}\delta_{\beta\zeta}\delta_{\gamma\theta})$

$$\mathrm{HO3_2} = -f_{n_1l_1j_1}(q^2) I^{(1)\,0}_{n_1l_1j_1,n_1l_1j_1} \sum_L (-1)^L (2L+1) (\bar{I}^{(2)\,L}_{n_2l_2j_2,n_3l_3j_3})^2\,;$$

* *term 3.3* $(\delta_{\alpha\theta}\delta_{\beta\delta}\delta_{\gamma\zeta})$ is equal to *term 3.6* $(\delta_{\alpha\zeta}\delta_{\beta\theta}\delta_{\gamma\delta})$

$$\mathrm{HO3_3} = \mathrm{HO3_6} = -f_{n_1l_1j_1,\,n_2l_2j_2}(q^2) I^{(2)\,0}_{n_3l_3j_3,n_3l_3j_3} \times \sum_L (2L+1) \bar{I}^{(1)\,L}_{n_1l_1j_1,n_2l_2j_2}\bar{I}^{(2)\,L}_{n_1l_1j_1,n_2l_2j_2}\,;$$

* *term 3.4* $(\delta_{\alpha\theta}\delta_{\beta\zeta}\delta_{\gamma\delta})$ is equal to *term 3.5* $(\delta_{\alpha\zeta}\delta_{\beta\delta}\delta_{\gamma\theta})$

$$\mathrm{HO3_4} = \mathrm{HO3_5} = -f_{n_1l_1j_1,\,n_3l_3j_3}(q^2) \times \sum_{L_2}(2L_2+1)\sum_{L_3}(2L_3+1)\sum_{L_1} i^{L_1+L_2+L_3} \times \sqrt{2L_1+1}\langle L_2 0\, L_3 0\,|\,L_1 0\rangle \begin{Bmatrix} \mathrm{L_3} \\ \mathrm{L_1} \\ \mathrm{L_2} \\ \mathrm{j_1} \\ \mathrm{j_2} \\ \mathrm{j_3} \end{Bmatrix} \times \bar{I}^{(1)\,L_1}_{n_1l_1j_1,n_3l_3j_3}\bar{I}^{(2)\,L_2}_{n_2l_2j_2,n_1l_1j_1}\bar{I}^{(2)\,L_3}_{n_3l_3j_3,n_2l_2j_2}\,;$$

In Figs. **??** and **??** we illustrate the convergence of the many-body expansion for the case of the ^{4}He and ^{16}O nuclei, respectively. The solid line represents the *exact* form factor in the center-of-mass system, as given by Eq. (**??**). The agreement is excellent for a momentum transfer $q < 3$ fm^{-1} and remains reasonably good for q up to 4 fm^{-1}. It is expected that the size of the contributions due to correlations (as presented in the next section) is more important than the error made by ignoring higher-order terms in the many-body expansion (**??**). Also, it is worthwhile mentioning that a correction expected to become increasingly important for high values of the momentum transfer is the contribution due to the meson-exchange charge density [**?**]. However, the inclusion of this correction is beyond the purpose of the present discussion.

We conclude that truncating the calculation at the third-order gives us a good approximation of the center-of-mass correction for the independent-particle model wave function case. Note that leaving out the three-body term in the case of the ^{4}He nucleus would result in an unacceptable description of the form factor distribution – *false* minima are located at a momentum transfer q as low as 3.6 fm^{-1} – whereas in the case of the ^{16}O nucleus, the charge form factor changes very little by including the three-body term. This is an indication that expression (**??**) can be viewed effectively as a $1/A$ power expansion of the charge form factor. Therefore, as we consider the applicability of the expansion (**??**) for higher values of A, it appears that we can safely drop higher-order terms in the many-body expansion and still hope for a good description of the charge form factor.

To conclude our study of the convergence of the many-body expansion, let us investigate the influence a given order of approximation has on the inferred mean square charge (rms) radius. It is well known that in the low q limit, the form factor may be be expanded in a power series as

$$F_{int}(q) \;=\; 1 \,-\, \frac{1}{6}\,q^2\,\langle r^2\rangle \,+\, \cdots\,,$$

and thus is a measure of the rms radius. Table **??** shows the convergence of the rms radius for the case of the ^{4}He and ^{16}O nuclei. These results show that the rms radius

is little affected by any corrections beyond the two-body term of the expansion (**??**). By including the three-body term in Eq. (**??**), the rms radius remains virtually the same in the ^{4}He case, and changes by less than 1% in the ^{16}O case.

IV. REALISTIC NUCLEAR WAVE FUNCTION USING THE $\exp(S)$ METHOD

We shall apply now our formalism to the case of a more complicated model wave function $\Phi_0^{(M)}$ and the particular case of the ^{16}O nucleus. As advertised, the nuclear wave function $\Phi_0^{(M)} = |\tilde{0}\rangle$, has been obtained using the coupled cluster method (or the exp(**S**) method) together with a *realistic* interaction [**?**]. The *exact* correlated ground state ket wave function $|\tilde{0}\rangle$ is written in terms of the uncorrelated ground state $|0\rangle$ as

$$|\tilde{0}\rangle = e^{\mathbf{S}^\dagger}|0\rangle \, . \tag{4.1}$$

Here, S† is the cluster correlation operator, which may be decomposed in terms of *ph*-creation operators ($\mathbf{O}_0^\dagger = \mathbf{1}$, $\mathbf{O}_1^\dagger = \mathbf{a}_{p_1}^\dagger \mathbf{a}_{h_1}$, $\mathbf{O}_2^\dagger = \mathbf{a}_{p_1}^\dagger \mathbf{a}_{p_2}^\dagger \mathbf{a}_{h_2} \mathbf{a}_{h_1}$) as

$$\mathbf{S}^\dagger = \sum_{n=0}^{\infty} \frac{1}{n!} S_n \mathbf{O}_n^\dagger \, .$$

The expectation value of an arbitrary operator A in the energy eigenstate (**??**) may be written as

$$\bar{A} = \langle 0 \,|\, e^{\mathbf{S}} \, A \, e^{-\mathbf{S}} \, \tilde{\mathbf{S}}^\dagger \,|\, 0 \rangle \, ,$$

where similarly to $\mathbf{S}^\dagger$, $\tilde{\mathbf{S}}^\dagger$ is defined by its decomposition in terms of *ph*-creation operators

$$\tilde{\mathbf{S}}^\dagger = \sum_{n} \frac{1}{n!} \tilde{S}_n \mathbf{O}_n^\dagger \, .$$

Therefore, the correct translation-invariant form factor is given by the expectation value of the operator F_{int} in the correlated ground state $|\tilde{0}\rangle$. As we have previously [**?**] worked out the one- and two-body densities for the ground state, we can apply these results to evaluate the first two terms in this expansion.

Using the definition of the one-body density

$$\rho(\vec{r}) \;=\; \sum_m \langle \tilde{0} \,|\, \delta(\vec{r} - \vec{r}_m) \,|\, \tilde{0} \rangle \, ,$$

together with the identity

$$e^{i\vec{q}\cdot\vec{r}_k} \;=\; \int d\vec{r} \; e^{i\vec{q}\cdot\vec{r}} \; \delta(\vec{r} - \vec{r}_k) \, ,$$

we can write the first term of Eq. (**??**) as

$$A_1 = f_p(q^2) \int d\vec{r} \; e^{i\vec{q}\cdot\vec{r}(A-1)/A} \; \rho^{(p)}(\vec{r}) + f_n(q^2) \int d\vec{r} \; e^{i\vec{q}\cdot\vec{r}(A-1)/A} \; \rho^{(n)}(\vec{r}) \, .$$

Here, $\rho^{(p)}(\vec{r})$ and $\rho^{(n)}(\vec{r})$ are the proton and neutron ground state one-body densities, which include corrections due to $2p2h$, $3p3h$, and $4p4h$ correlations.

Similarly, we can write the second term as a double integral over the ground state two-body density using

$$\rho(\vec{r}_1, \vec{r}_2) \;=\; \sum_{mn} \langle \tilde{0} \,|\, \delta(\vec{r}_1 - \vec{r}_m) \; \delta(\vec{r}_2 - \vec{r}_n) \,|\, \tilde{0} \rangle \, .$$

Then, the second term of Eq. (**??**) becomes

$$\begin{aligned} A_2 = f_p(q^2) \int d\vec{r} \int d\vec{r'} \; e^{i\vec{q}\cdot\vec{r}(A-1)/A} \; f^*(\vec{q}\cdot\vec{r}'/A) \\ \Big[\rho^{(p,p)}(\vec{r},\vec{r}') + \rho^{(p,n)}(\vec{r},\vec{r}')\Big] \\ + f_n(q^2) \int d\vec{r} \int d\vec{r'} \; e^{i\vec{q}\cdot\vec{r}(A-1)/A} \; f^*(\vec{q}\cdot\vec{r}'/A) \\ \Big[\rho^{(n,p)}(\vec{r},\vec{r}') + \rho^{(n,n)}(\vec{r},\vec{r}')\Big] \, . \end{aligned}$$

With these evaluations we include all the terms that were included in evaluating the one- and two-body densities.

Let us note here that the approximation (A_1+A_2) is insufficient in order to get an exact value of the rms radius. Fortunately, we find that the three-body density contributes to the q^2-dependence of the form factor only through the term in which $L_1 = 0$ (see Eq. (**??**)). Thus, in order to get the proper low q dependence exactly up to powers of q^2, we only need to include the term

$$\begin{aligned} A_{3p} \;=\; A_1(q) \int d\vec{r} \int d\vec{r'} \; f^*(\vec{q}\cdot\vec{r}/A) \, f^*(\vec{q}\cdot\vec{r}'/A) \\ \times \Big[\rho^{(p,p)}(r,r') + \rho^{(p,n)}(r,r') + \rho^{(n,p)}(r,r') + \rho^{(n,n)}(r,r')\Big] \, . \end{aligned}$$

We conclude that the one- and two-body densities are sufficient to calculate an exact rms radius by using the approximation (A_1+A_2+A_{3p}). However, one should not attempt to use A_{3p} indiscriminately as a charge form factor correction for an arbitrary value of q, since A_{3p} is not a consistent approximation of the three-body term, A_3, in Eq. (**??**). Any attempt of including A_3, by an approximation short of a consistent approximation of the three-body density itself, will introduce uncontrollable errors in the calculation of the charge form factor. Therefore, A_{3p} will be considered only in the calculation of the rms radius, which then typically changes by less than 4%. This seems to indicate that the correction due to the three-body term in Eq. (**??**) is, in general, small, hopefully smaller than other effects we leave out at this time (like the contributions due to the meson-exchange charge density), and shall henceforth be ignored in the calculation of the charge form factor.

V. RESULTS AND CONCLUSIONS

The problem of center-of-mass corrections in calculating observables has been worked out by expanding the center-of-mass correction as many-body operators. We have applied this expansion to the case of the harmonic oscillator where an exact solution exists. We found reasonable convergence in the case of harmonic oscillator wave functions. Thus we have confidence that this method can be applied to general Hartree-Fock wave functions and in a situation where $2p2h$-correlations are present.

Figures **??** and **??** show the various effects of the correlations on the internal charge form factor, corresponding to calculations using the Argonne $v18$ together with the Urbana-IX potential. We also compare the various approximations of the form factor with the internal form factor suggested by Eq. (**??**), which in both cases is plotted as a dotted line.

In the calculation of the translational invariant charge form factor, correlations enter at two places. First, the calculation of the one-body operator (A1) includes effects of all the correlations, because this term is simply the Fourier transform of the one-body density. In Fig. **??**, the solid and dashed lines represent the Fourier transform of the one-body density corresponding to the uncorrelated ($|0\rangle$) and correlated ($|\tilde{0}\rangle$) ground state, respectively. These form factors are denoted $SM1[\rho_0(r)]$ and $SM1[\rho(r)]$. Here, the main effect of the correlations is the shifting of the diffraction minimum by 5% to the right. The new minimum is also predicted by Eq. (**??**), which also has a higher tail compared to $SM1[\rho_0(r)]$ and $SM1[\rho(r)]$.

Secondly, as any expectation value taken in the correlated ground state, the center-of mass corrections are modified due to the correlations. In Fig. **??**, the solid and dashed lines represent the two-body approximations of the translational invariant form factor. Going beyond the leading order ($SM2$) in evaluating the two-body term ($A2$) leaves the first diffraction minimum virtually unchanged. However, the high q behavior of the form factor, ($q > 2.5\mathrm{fm}^{-1}$), is dramatically affected. We can see that the (A_1+A_2) approximation of the internal charge form factor exhibits a second diffraction minimum, which has been observed experimentally by Sick and McCarthy [**?**], and its presence makes our theory credible. Physically speaking, the hole in the two-body density affects the center-of-mass motion and thus the center-of-mass correction to be applied.

ACKNOWLEDGEMENTS

This work was supported in part by the U.S. Department of Energy (DE-FG02-87ER-40371). The work of B.M. was also supported in part by the U.S. Department of Energy under contract number DE-FG05-87ER40361 (Joint Institute for Heavy Ion Research), and DE-AC05-96OR22464 with Lockheed Martin Energy Research Corp. (Oak Ridge National Laboratory). The calculations were carried out on a dual-processor 500 MHz Pentium II PC in the Nuclear Physics Group at the University of New Hampshire, and a 180 MHz R10000 Silicon Graphics Workstation in the Computational and Theoretical Physics Section at ORNL. The authors gratefully acknowledge the help of Mike Strayer and David Dean in letting us use their computing resources. The authors also gratefully acknowledge John Dawson for his permanent support in every step of this project.

APPENDIX: NOTATIONS

We present here the various notations used in the text. We have

$$
\begin{aligned}
I^{(1)\,L}_{n_1l_1j_1,n_2l_2j_2} &= (2j_1\,\delta_{(n_1l_1j_1),(n_2l_2j_2)} + 1) \\
&\times \int_0^\infty \mathcal{HO}_{n_1l_1}(r)\mathcal{HO}_{n_2l_2}(r) j_L(qr(A-1)/A)\, r^2 dr\,. \\
I^{(2)\,L}_{n_1l_1j_1,n_2l_2j_2} &= (2j_1\,\delta_{(n_1l_1j_1),(n_2l_2j_2)} + 1) \\
&\times \int_0^\infty \mathcal{HO}_{n_1l_1}(r)\mathcal{HO}_{n_2l_2}(r) f_L(qr/A)\, r^2 dr\,,
\end{aligned}
$$

where $\mathcal{HO}_{nl}(r)$ are the usual radial harmonic oscillator wave functions. The symbol $\delta_{(n_1l_1j_1),(n_2l_2j_2)}$ is one where the set of indices $(n_1l_1j_1)$ and $(n_2l_2j_2)$ represent the same single-particle wave function, and zero otherwise. We also introduce

$$
\bar{I}^{(1,2)\,L}_{n_1l_1j_1,n_2l_2j_2} = \langle (l_1\tfrac{1}{2})j_1 \,||\, C^{(L)} \,||\, (l_2\tfrac{1}{2})j_2\rangle\; I^{(1,2)\,L}_{n_1l_1j_1,n_2l_2j_2}\,.
$$

The reduced matrix element of the unnormalized spherical harmonic operator of rank k is

$$
\begin{aligned}
&\left\langle (l_a\tfrac{1}{2})\,j_a \,\middle\|\, C^{(k)} \,\middle\|\, (l_b\tfrac{1}{2})\,j_b \right\rangle \\
&= (-1)^{j_a+k+\frac{3}{2}} \sqrt{\frac{(2j_a+1)(2j_b+1)}{(2k+1)}}\, \langle j_a\ \tfrac{1}{2}\ j_b\ \tfrac{1}{2}\ \mid k\,0\rangle
\end{aligned}
$$

for $|l_1 - l_2| \le k \le l_1 + l_2$ and $|j_1 - j_2| \le k \le j_1 + j_2$, and zero otherwise.

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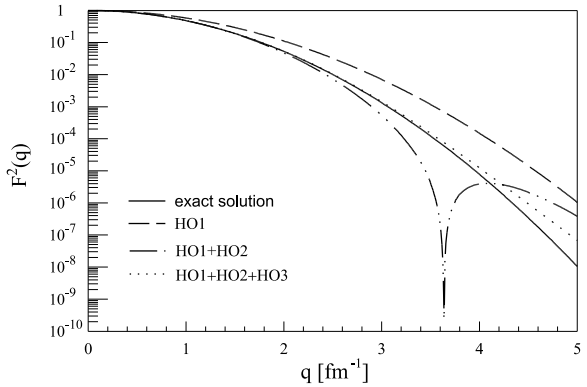

FIG. 1. Convergence of the many-body expansion (2.4) of the charge form factor for the harmonic oscillator shell-model case and a ^{4}He-like nucleus.

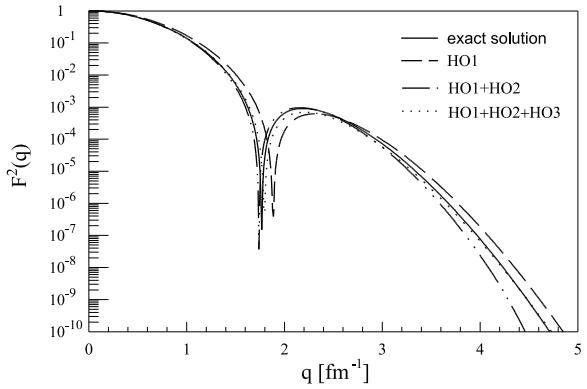

FIG. 2. Same as Fig. 1, except for a ^{16}O-like nucleus.

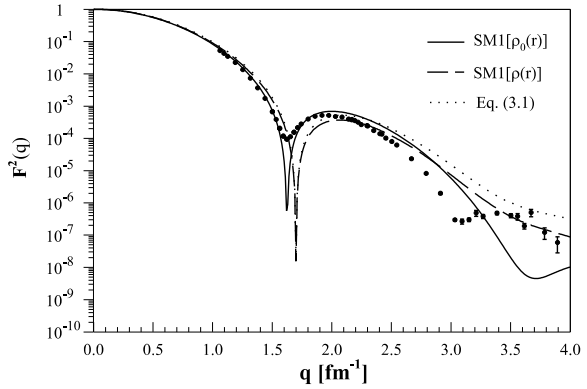

FIG. 3. ^{16}O nucleus: $SM1[\rho_0(r)]$ and $SM1[\rho(r)]$ form factors compared with the internal form factor calculated according to Eq. (3.1) for the Argonne $v18$ and Urbana-IX potentials.

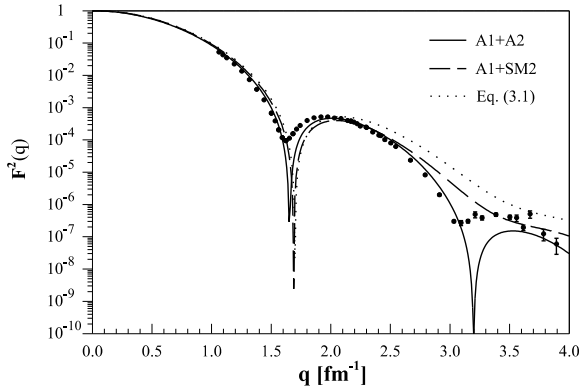

FIG. 4. ^{16}O nucleus: Two-body approximations of the translational invariant form factor compared with the internal form factor calculated according to Eq. (3.1) for the Argonne $v18$ and Urbana-IX potentials.

TABLE I. Convergence of the mean square charge radius for the case of the ^{4}He and ^{16}O-like nuclei.

Order of approximation	^{4}He	^{16}O
HO1	1.285979	2.250000
HO1 + HO2	1.484927	2.371708
HO1 + HO2 + HO3	1.484922	2.349467
exact value	1.484924	2.349468