Tensor-decomposition of three-nucleon interaction

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This is the working paper to derive matrix elements for the new three-nucleon interaction proposed by Pieper, Pandharipande, Wiringa, and Carlson. All the matrix elements contain the "Ring"-phase which is explicitly included by using the **type (a)** and **type (b)** matrix elements as specified below.

First we note that the two terms of the Urbana IX potential are the same, except for modifies coefficients which are the Fujita - Miyasawa term and the short range repulsion. There are two new terms, the two-pion S-wave term and the three-pion Δ term. We devote section 1 for the S-wave term and section 2 for the latter.

Similar to our derivation of the Urbana IX matrix elements we focus on two type of matrix elements given in (3.2) of that report:

type (a)

$$\langle(a\bar{b})_\lambda|V^{eff}|(c\bar{d})_\lambda\rangle = \delta_{m_h,m_p}(-)^{k_a+k_b+k_c+k_d}(-)^{m_b-m_d} \langle j_a m_a j_b - m_b|\lambda\mu\rangle\langle j_c m_c j_d - m_d|\lambda\mu\rangle V_{m_h m_a m_d, m_p m_b m_c} \tag{0.1}$$

With a tensor operator coupled in the form:

$$\left(\sqrt{4\pi}\right)^3\Big[T_1^{(\lambda_1)}(1)\otimes\big[T_2^{(\lambda_2)}(2)\otimes T_3^{(\lambda_3)}(3)\big]^{(\lambda_1)}\Big]^{(0)} \tag{0.2}$$

The **type (a)** matrix element requires that $T_1^{(\lambda_1)} = Y^{(0)}$, as only this gives a non-vanishing contribution. This allows the interaction being brought into the form

$$V = 4\pi\big[\bar{T}_1^\lambda \odot \bar{T}_2^\lambda\big] \tag{0.3}$$

Then, using (2.1) from the V-18 write-up, the matrix element including the "Ring"-phase is given as

$$\langle p_1\bar{h}_1|V^{eff,\lambda}|h_2\bar{p}_2\rangle = \int r_i^2 dr_i R_h(r_i)R_p(r_i) \times\Big[\frac{(-)^{k_{p_1}}\sqrt{4\pi}}{\hat{\lambda}}\langle p_1||\bar{T}_1^\lambda||h_1\rangle\Big]\Big[\frac{(-)^{k_{p_2}}\sqrt{4\pi}}{\hat{\lambda}}\langle p_2||\bar{T}_2^\lambda||h_2\rangle\Big] \tag{0.4}$$

Those terms that depend on all three σ's do not contribute here. The terms that depend on two σ's need to be split into the three types:

σ type

$$\langle p_1\bar{h}_1|V^{eff,\sigma,\lambda}|h_2\bar{p}_2\rangle = K^{eff,\ell}(r_j,r_k) \times(-)^{(\ell+1+\lambda)}\Big[\frac{(-)^{k_{p_1}}\sqrt{4\pi}}{\hat{\lambda}}\langle p_1||[Y^\ell\sigma_j]^\lambda||h_1\rangle\Big]\Big[\frac{(-)^{k_{p_2}}\sqrt{4\pi}}{\hat{\lambda}}\langle p_2||[Y^\ell\sigma_k]^\lambda||h_2\rangle\Big] \tag{0.5}$$

tensor type

$$\langle p_1\bar{h}_1|V^{eff,\sigma,\lambda}|h_2\bar{p}_2\rangle = K^{eff,\ell_1,\ell_2}(r_j,r_k)\ \sqrt{6}\hat{\ell}_1\hat{\ell}_2\langle\ell_1 0\ell_2 0|20\rangle(i)^{\ell_1+\ell_2}\begin{Bmatrix}1 & 1 & 2\\ \ell_1 & \ell_2 & \lambda\end{Bmatrix} \times\Big[\sqrt{4\pi}\frac{(-)^{k_{p_1}}}{\hat{\lambda}}\langle p_1||[Y^{\ell_1}\sigma_j]^\lambda||h_1\rangle\Big]\Big[\sqrt{4\pi}\frac{(-)^{k_{p_2}}}{\hat{\lambda}}\langle p_2||[Y^{\ell_2}\sigma_k]^\lambda||h_2\rangle\Big] \tag{0.6}$$

k=1 type

$$\langle p_1\bar{h}_1|V^{eff,\sigma,\lambda}|h_2\bar{p}_2\rangle = K^{eff,\ell}(r_j,r_k)\begin{Bmatrix}1 & 1 & 1\\ \ell & \ell & \lambda\end{Bmatrix} \times\Big[\frac{(-)^{k_{p_1}}\sqrt{4\pi}}{\hat{\lambda}}\langle p_1||[Y^\ell\sigma_j]^\lambda||h_1\rangle\Big]\Big[\frac{(-)^{k_{p_2}}\sqrt{4\pi}}{\hat{\lambda}}\langle p_2||[Y^\ell\sigma_k]^\lambda||h_2\rangle\Big] \tag{0.7}$$

type (b)

$$\langle (a\bar{b})_\lambda | V^{eff,X} | (c\bar{d})_\lambda \rangle = -\delta_{m_h,m_p} (-)^{k_a+k_b+k_c+k_d} (-)^{m_b-m_d} \\ \langle j_a m_a j_b - m_b | \lambda\mu \rangle \langle j_c m_c j_d - m_d | \lambda\mu \rangle V_{m_h m_a m_d, m_b m_p m_c} \tag{0.8}$$

With a tensor operator coupled in the form:

$$\left(\sqrt{4\pi}\right)^3 \Big[T_1^{(\lambda_1)}(1) \otimes \big[T_2^{(\lambda_2)}(2) \otimes T_3^{(\lambda_3)}(3)\big]^{(\lambda_1)}\Big]^{(0)} \tag{0.9}$$

we calculate the **type (b)** matrix element using (4.4) of the vtni paper

$$\begin{aligned} \langle p_1 \bar{h}_1 | V^{X,\lambda} | h_2 \bar{p}_2 \rangle = (-)^{(k_{p_1}+k_{h_1}+k_h+\lambda_1+\lambda_2+\lambda)} \frac{\hat{\lambda}_1 \hat{\lambda}_2}{\hat{\lambda}} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda \\ j_{p_1} & j_{h_1} & j_h \end{Bmatrix} \\ \times \Big\{ \Big[\frac{(-)^{k_h}\sqrt{4\pi}}{\hat{\lambda}_1} \langle h || T_1^{\lambda_1} || h_1 \rangle\Big] \Big[\frac{(-)^{k_{p_1}}\sqrt{4\pi}}{\hat{\lambda}_2} \langle p_1 || T_2^{\lambda_2} || h \rangle\Big] \Big[\frac{(-)^{k_{p_2}}\sqrt{4\pi}}{\hat{\lambda}} \langle p_2 || T_3^{\lambda} || h_2 \rangle\Big] \\ + \Big[\frac{(-)^{k_{p_1}}\sqrt{4\pi}}{\hat{\lambda}_2} \langle p_1 || T_1^{\lambda_2} || h \rangle\Big] \Big[\frac{(-)^{k_{p_2}}\sqrt{4\pi}}{\hat{\lambda}} \langle p_2 || T_2^{\lambda} || h_2 \rangle\Big] \Big[\frac{(-)^{k_h}\sqrt{4\pi}}{\hat{\lambda}_1} \langle h || T_3^{\lambda_1} || h_1 \rangle\Big] \\ + \Big[\frac{(-)^{k_{p_2}}\sqrt{4\pi}}{\hat{\lambda}} \langle p_2 || T_1^{\lambda} || h_2 \rangle\Big] \Big[\frac{(-)^{k_h}\sqrt{4\pi}}{\hat{\lambda}_1} \langle h || T_2^{\lambda_1} || h_1 \rangle\Big] \Big[\frac{(-)^{k_{p_1}}\sqrt{4\pi}}{\hat{\lambda}_2} \langle p_1 || T_3^{\lambda_2} || h \rangle\Big] \Big\} \end{aligned} \tag{0.10}$$

In addition to these two, we need the (pp)-coupled matrix elements formed from these two types:

$$\langle (a,b)_K | V^{eff} | (c,d)_K \rangle = (-)^{K+1} \sum_\lambda (2\lambda+1) \begin{Bmatrix} j_a & j_b & K \\ j_d & j_c & \lambda \end{Bmatrix} \langle a\bar{c} | V^{eff,\lambda} | d\bar{b} \rangle \tag{0.11}$$

Before we go into the details we work out some operator identities:

$$\vec{r}_{ij} = \sqrt{\frac{4\pi}{3}} r_{ij} Y^{(1)}(\hat{r}_{ij}) \tag{0.12}$$

Scalar or vector products are converted as

$$\vec{A} \cdot \vec{B} \ = -\sqrt{3}\big[A^{(1)} \otimes B^{(1)}\big]^{(0)} \tag{0.13}$$

$$\big[A^{(k)} \odot B^{(k)}\big] = (-)^k \hat{k} \big[A^{(k)} \otimes B^{(k)}\big] \tag{0.14}$$

$$\vec{A} \times \vec{B} \ = i\sqrt{2}\big[A^{(1)} \otimes B^{(1)}\big]^{(1)} \tag{0.15}$$

Spherical harmonics are combined as

$$\big[Y^{(k_1)}(\hat{r}) \otimes Y^{(k_2)}(\hat{r})\big]^{(k_3)}_\mu = \frac{\hat{k}_1 \ \hat{k}_2}{\sqrt{4\pi} \ \hat{k}_3} \langle k_1 0 k_2 0 | k_3 0 \rangle Y^{(k_3)}_\mu(\hat{r}) \tag{0.16}$$

We use the basic equation (2.2) from V-18 write-up:

$$V(r_{12}) Y^{(J)}(\hat{r}_{12}) = \frac{2}{\pi} \int q^2 dq \ \tilde{V}^J(q) \sum_{\ell_1,\ell_2} \frac{\hat{\ell}_1 \hat{\ell}_2}{\hat{J}} \frac{1}{\sqrt{4\pi}} \langle \ell_1 0 \ell_2 0 | J 0 \rangle \\ \times j_{\ell_1}(qr_1) j_{\ell_2}(qr_2) (i)^{(\ell_1-\ell_2-J)} \ 4\pi \ \big[Y^{(\ell_1)}(\hat{r}_1) \otimes Y^{(\ell_2)}(\hat{r}_2)\big]^{(J)}$$

with

$$\tilde{V}^J(q) = \int V(r) j_J(qr) r^2 dr$$

The integration over q can now be expressed as the kernel

$$T^{V,J,\ell_1,\ell_2}_{n,m} = \sum_i \; H^{\ell_1}_n(q_i) H^{\ell_2}_m(q_i) \tilde{V}^J(q_i) q_i^2 w_i \tag{0.17}$$

$$F^{J,\ell_1,\ell_2}_V(r_i,r_j) = \sum_{n,m} H^{\ell_1}_n(r_i) T^{V,J,\ell_1,\ell_2}_{n,m} H^{\ell_2}_m(r_j)$$

$$K^{J,\ell_1,\ell_2}_V(r_i,r_j) = \langle \ell_1 0 \ell_2 0 | J0\rangle (i)^{(\ell_1-\ell_2-J)} \frac{\hat{\ell}_1\hat{\ell}_2}{\hat{J}} F^{J,\ell_1,\ell_2}_V(r_i,r_j) \tag{0.18}$$

which has the symmetry

$$K^{J,\ell_1,\ell_2}_V(r_i,r_j) = (-)^{(\ell_1-\ell_2)} K^{J,\ell_2,\ell_1}_V(r_j,r_i) \tag{0.19}$$

Finally,

$$V(r_{ij}) Y^{(J)}(\hat{r}_{ij}) = \; \sqrt{4\pi} \sum_{\ell_1,\ell_2} K^{J,\ell_1,\ell_2}_V(r_i,r_j) \; \left[Y^{(\ell_1)}(\hat{r}_i) \otimes Y^{(\ell_2)}(\hat{r}_j) \right]^{(J)}$$

I. P-WAVE TWO-PION-EXCHANGE

We take the two-pion exchange interaction from our paper on V-tni (eqn.2.29) as:

$$\begin{aligned} V^{2\pi,P,C}(\vec{r}_i,\vec{r}_j,\vec{r}_k) = 9\big(\sqrt{4\pi}\big)^3 \sum_{K_2,K_3} \langle 1010|K_2 0\rangle \langle 1010|K_3 0\rangle K^{K_2,\ell_1,\ell_2}_{2\pi}(r_i,r_j) K^{K_3,\ell_3,\ell_4}_{2\pi}(r_i,r_k) \langle \ell_1 0 \ell_4 0 | L0\rangle \\ \times (-)^{(\ell_2+\lambda_2+\ell_3+\lambda_3)} \hat{K}_2\hat{K}_3\hat{\lambda}_2\hat{\lambda}_3\hat{\ell}_1\hat{\ell}_4\hat{\lambda}_1\hat{S} \begin{Bmatrix} \ell_1 & 1 & \lambda_2 \\ \ell_4 & 1 & \lambda_3 \\ L & S & \lambda_1 \end{Bmatrix} \begin{Bmatrix} \ell_1 & \ell_2 & K_2 \\ 1 & 1 & \lambda_2 \end{Bmatrix} \begin{Bmatrix} \ell_4 & \ell_3 & K_3 \\ 1 & 1 & \lambda_3 \end{Bmatrix} \\ \times \left[\left[Y^{(L)}(\hat{r}_i) \otimes \left[\sigma^{(1)}_i \otimes \sigma^{(1)}_i\right]^{(S)} \right]^{(\lambda_1)} \otimes \left[\left[Y^{(\ell_2)}(\hat{r}_j) \otimes \sigma^{(1)}_j \right]^{(\lambda_2)} \otimes \left[Y^{(\ell_3)}(\hat{r}_k) \otimes \sigma^{(1)}_k \right]^{(\lambda_3)} \right]^{(\lambda_1)} \right]^{(0)} \end{aligned} \tag{1.1}$$

In segment C we deal with the Commutator-term with a strength of $A_{2\pi}/4$. In segment A and B we deal with the anti-commutator term which requires $S=0$. Here we use that $[\sigma\otimes\sigma]^{(0)} = -\sqrt{3}$. After evaluating the 9-j-symbol and using a factor 2 from the anti-commutator and a factor 2 from the τ-anti-commutator we find the interaction as:

$$\begin{aligned} V^{2\pi,P,C}(\vec{r}_i,\vec{r}_j,\vec{r}_k) = 36\; A_{2\pi}\; \big(\sqrt{4\pi}\big)^3 \sum_{K_2,K_3} \langle 1010|K_2 0\rangle \langle 1010|K_3 0\rangle K^{K_2,\ell_1,\ell_2}_{2\pi}(r_i,r_j) K^{K_3,\ell_3,\ell_4}_{2\pi}(r_i,r_k) \langle \ell_1 0 \ell_4 0 | L0\rangle \\ \times (-)^{(\ell_2+\lambda_2+\ell_3+\lambda_1+\ell_1)} \hat{K}_2\hat{K}_3\hat{\lambda}_2\hat{\lambda}_3\hat{\ell}_1\hat{\ell}_4 \begin{Bmatrix} \lambda_2 & \ell_1 & 1 \\ \ell_4 & \lambda_3 & \lambda_1 \end{Bmatrix} \begin{Bmatrix} \ell_1 & \ell_2 & K_2 \\ 1 & 1 & \lambda_2 \end{Bmatrix} \begin{Bmatrix} \ell_4 & \ell_3 & K_3 \\ 1 & 1 & \lambda_3 \end{Bmatrix} \\ \times \left[Y^{(\lambda_1)}(\hat{r}_i) \otimes \left[\left[Y^{(\ell_2)}(\hat{r}_j) \otimes \sigma^{(1)}_j \right]^{(\lambda_2)} \otimes \left[Y^{(\ell_3)}(\hat{r}_k) \otimes \sigma^{(1)}_k \right]^{(\lambda_3)} \right]^{(\lambda_1)} \right]^{(0)} \langle \vec{\tau}_j \vec{\tau}_k \rangle \end{aligned} \tag{1.2}$$

A) P-WAVE ANTI-COMMUTATOR TERM

The **type (a)** matrix element is derived from the above interaction setting $L=0$. From that it follows that $\lambda_1 = 0$ and $\ell_1 = \ell_4$ as well as $\lambda_3 = \lambda_2 =: \lambda$. We also convert the K's into F's and find

$$\begin{aligned} V^{2\pi,P,C}(\vec{r}_i,\vec{r}_j,\vec{r}_k) = A_{2\pi}\; 36\big(4\pi\big) \sum_{K_2,K_3} \langle 1010|K_2 0\rangle \langle 1010|K_3 0\rangle K^{K_2,\ell_1,\ell_2}_{2\pi}(r_i,r_j) K^{K_3,\ell_3,\ell_1}_{2\pi}(r_i,r_k) \\ \times (-)^{\lambda+1+\ell_1} \hat{K}_2\hat{K}_3 \begin{Bmatrix} \ell_1 & \ell_2 & K_2 \\ 1 & 1 & \lambda \end{Bmatrix} \begin{Bmatrix} \ell_1 & \ell_3 & K_3 \\ 1 & 1 & \lambda \end{Bmatrix} \\ \times \left[\left[Y^{(\ell_2)}\sigma_j \right]^{(\lambda)} \odot \left[Y^{(\ell_3)}\sigma_k \right]^{(\lambda)} \right] \langle \vec{\tau}_j \vec{\tau}_k \rangle \end{aligned} \tag{1.3}$$

or

$$V^{2\pi,P,C}(\vec{r}_i,\vec{r}_j,\vec{r}_k) = A_{2\pi}\ 36\sum_{K_2,K_3}\langle 1010|K_20\rangle\langle 1010|K_30\rangle F_{2\pi}^{K_2,\ell_1,\ell_2}(r_i,r_j)F_{2\pi}^{K_3,\ell_3,\ell_1}(r_i,r_k)$$
$$\times(-)^{\lambda+1+\ell_2}\langle\ell_10\ell_20|K_20\rangle\langle\ell_10\ell_30|K_30\rangle\left\{\begin{array}{ccc}\ell_1 & \ell_2 & K_2\\ 1 & 1 & \lambda\end{array}\right\}\left\{\begin{array}{ccc}\ell_1 & \ell_3 & K_3\\ 1 & 1 & \lambda\end{array}\right\}$$
$$(i)^{(\ell_2-\ell_3+K_2-K_3)}\hat{\ell}_2\hat{\ell}_3(2\ell_1+1)$$
$$\times(4\pi)\left[\left[Y^{(\ell_2)}\sigma_j\right]^{(\lambda)}\odot\left[Y^{(\ell_3)}\sigma_k\right]^{(\lambda)}\right]\ \langle\vec{\tau}_j\vec{\tau}_k\rangle \tag{1.4}$$

We treat the four resulting cases separately starting with

case K_2=0,K_3=0

This requires $\ell_1=\ell_2=:\ell$ and $\ell_1=\ell_3=\ell$.

$$V^{2\pi,P,C}(\vec{r}_i,\vec{r}_j,\vec{r}_k) = A_{2\pi}\ 4F_{2\pi}^{0,\ell,\ell}(r_i,r_j)F_{2\pi}^{0,\ell,\ell}(r_i,r_k)$$
$$\times(-)^{\ell+\lambda+1}\left[\left[\sqrt{4\pi}Y^{(\ell)}\sigma_j\right]^{(\lambda)}\odot\left[\sqrt{4\pi}Y^{(\ell)}\sigma_k\right]^{(\lambda)}\right]\ \langle\vec{\tau}_j\vec{\tau}_k\rangle \tag{1.5}$$

This term is added to the $\sigma\sigma$-interaction.

cases K_2=0,K_3=2, and K_2=2,K_3=0

Both those cases add to the tensor interaction.

defining:

$$tl^{\lambda,\ell_1,\ell_2} = \sqrt{6}\ \hat{\ell}_1\hat{\ell}_2\langle\ell_10\ell_20|20\rangle(i)^{\ell_1+\ell_2}\left\{\begin{array}{ccc}\ell_1 & \ell_2 & 2\\ 1 & 1 & \lambda\end{array}\right\} \tag{1.6}$$

we obtain

$$V^{2\pi,P,C}(\vec{r}_i,\vec{r}_j,\vec{r}_k) = A_{2\pi}\ 4\ F_{2\pi}^{0,\ell_2,\ell_2}(r_i,r_j)F_{2\pi}^{2,\ell_3,\ell_2}(r_i,r_k)\ tl^{\lambda,\ell_2,\ell_3}$$
$$\times\left[\left[\sqrt{4\pi}Y^{(\ell_2)}\sigma_j\right]^{(\lambda)}\odot\left[\sqrt{4\pi}Y^{(\ell_3)}\sigma_k\right]^{(\lambda)}\right]\ \langle\vec{\tau}_j\vec{\tau}_k\rangle \tag{1.7}$$

and

$$V^{2\pi,P,C}(\vec{r}_i,\vec{r}_j,\vec{r}_k) = A_{2\pi}\ 4\ F_{2\pi}^{2,\ell_2,\ell_3}(r_i,r_j)F_{2\pi}^{0,\ell_2,\ell_2}(r_i,r_k)\ tl^{\lambda,\ell_2,\ell_3}$$
$$\times\left[\left[\sqrt{4\pi}Y^{(\ell_3)}\sigma_j\right]^{(\lambda)}\odot\left[\sqrt{4\pi}Y^{(\ell_2)}\sigma_k\right]^{(\lambda)}\right]\ \langle\vec{\tau}_j\vec{\tau}_k\rangle \tag{1.8}$$

case K_2=2,K_3=2

This case will be split into three separate cases. Using

$$(-)^{j_1+j_2+j_1'+j_2'+j_1"+j_2"+J}\left\{\begin{array}{ccc}j_1 & j_2 & J\\ j_1" & j_2" & \lambda_1\end{array}\right\}\left\{\begin{array}{ccc}j_1' & j_2' & J\\ j_1" & j_2" & \lambda_2\end{array}\right\}$$
$$=\sum_k(-)^{\lambda_1+\lambda_2+k}(2k+1)\left\{\begin{array}{ccc}j_1 & j_1' & k\\ j_2' & j_2 & J\end{array}\right\}\left\{\begin{array}{ccc}\lambda_1 & \lambda_2 & k\\ j_1' & j_1 & j_2"\end{array}\right\}\left\{\begin{array}{ccc}\lambda_1 & \lambda_2 & k\\ j_2' & j_2 & j_1"\end{array}\right\} \tag{1.9}$$

with J=ℓ_1, j_1"=1 and j_2"=λ, λ_1=1, λ_2=1, j_1=ℓ_2, j_2=2, j_1'=ℓ_3, and j_2'= 2.

$$V^{2\pi,P,C}(\vec{r}_i,\vec{r}_j,\vec{r}_k) = A_{2\pi}\ 24F_{2\pi}^{2,\ell_1,\ell_2}(r_i,r_j)F_{2\pi}^{2,\ell_3,\ell_1}(r_i,r_k)\langle\ell_10\ell_20|20\rangle\langle\ell_10\ell_30|20\rangle\hat{\ell}_1\hat{\ell}_2\hat{\ell}_1\hat{\ell}_3$$
$$\times(i)^{(\ell_2-\ell_3)}(-)^{(\lambda+1+\ell_2)}\sum_k(-)^{\ell_1+\ell_2+\ell_3+\lambda+1+k}(2k+1)\left\{\begin{array}{ccc}\ell_2 & \ell_3 & k\\ 1 & 1 & \lambda\end{array}\right\}\left\{\begin{array}{ccc}\ell_2 & \ell_3 & k\\ 2 & 2 & \ell_1\end{array}\right\}\left\{\begin{array}{ccc}2 & 2 & k\\ 1 & 1 & 1\end{array}\right\}$$
$$\times\left[\left[\sqrt{4\pi}Y^{(\ell_2)}\sigma_j\right]^{(\lambda)}\odot\left[\sqrt{4\pi}Y^{(\ell_3)}\sigma_k\right]^{(\lambda)}\right]\ \langle\vec{\tau}_j\vec{\tau}_k\rangle \tag{1.10}$$

Here k is limited to the values k=0, 1, 2 due to selection rules in the first 6j-symbol. Here again, the $k=0$ term contributes to the $\sigma\sigma$ interaction, and the $k=2$ term contributes to the tensor interaction.

case k=0

$$V^{2\pi,P,C}(\vec{r}_i,\vec{r}_j,\vec{r}_k) = A_{2\pi}\ \frac{8}{5}\ F_{2\pi}^{2,\ell_1,\ell_2}(r_i,r_j)F_{2\pi}^{2,\ell_2,\ell_1}(r_i,r_k)\ (2\ell_1+1)$$
$$\times\ (-)^{\lambda+1+\ell_2}\left[\left[\sqrt{4\pi}Y^{(\ell_2)}\sigma_j\right]^{(\lambda)}\odot\left[\sqrt{4\pi}Y^{(\ell_2)}\sigma_k\right]^{(\lambda)}\right]\ \langle\vec{\tau}_j\vec{\tau}_k\rangle \tag{1.11}$$

case k=2

$$V^{2\pi,P,C}(\vec{r}_i,\vec{r}_j,\vec{r}_k) = A_{2\pi}\ 4\sqrt{\frac{7}{2}}\ F_{2\pi}^{2,\ell_1,\ell_2}(r_i,r_j)F_{2\pi}^{2,\ell_3,\ell_1}(r_i,r_k)\ tl^{\lambda,\ell_2,\ell_3}$$
$$\times(2\ell_1+1)(-)^{\ell_1}\left\{\begin{matrix}\ell_2 & \ell_3 & 2\\ 2 & 2 & \ell_1\end{matrix}\right\}\frac{\langle\ell_1 0\ell_2 0|20\rangle\langle\ell_1 0\ell_3 0|20\rangle}{\langle\ell_2 0\ell_3 0|20\rangle}$$
$$\times\left[\left[\sqrt{4\pi}Y^{(\ell_2)}\sigma_j\right]^{(\lambda)}\odot\left[\sqrt{4\pi}Y^{(\ell_3)}\sigma_k\right]^{(\lambda)}\right]\ \langle\vec{\tau}_j\vec{\tau}_k\rangle \tag{1.12}$$

case k=1

For the $k=1$ term the first 6j-symbol requires that $|\ell_2-\ell_3|\leq 1$ which together with $\ell_2+\ell_3$=even requires $\ell_2=\ell_3$.

$$V^{2\pi,P,C}(\vec{r}_i,\vec{r}_j,\vec{r}_k) = A_{2\pi}\ \frac{36}{\sqrt{5}}F_{2\pi}^{2,\ell_1,\ell_2}(r_i,r_j)F_{2\pi}^{2,\ell_2,\ell_1}(r_i,r_k)$$
$$\times(2\ell_1+1)(2\ell_2+1)\langle\ell_1 0\ell_2 0|20\rangle^2\left\{\begin{matrix}\ell_2 & \ell_2 & 1\\ 1 & 1 & \lambda\end{matrix}\right\}\left\{\begin{matrix}\ell_2 & \ell_2 & 1\\ 2 & 2 & \ell_1\end{matrix}\right\}$$
$$\times\left[\left[\sqrt{4\pi}Y^{(\ell_2)}\sigma_j\right]^{(\lambda)}\odot\left[\sqrt{4\pi}Y^{(\ell_2)}\sigma_k\right]^{(\lambda)}\right]\ \langle\vec{\tau}_j\vec{\tau}_k\rangle \tag{1.13}$$

MORE ANTI-COMMUTATOR TERMS

The **type b** matrix elements no longer have the restriction of $L=0$.

$$V^{2\pi,P,C}(\vec{r}_i,\vec{r}_j,\vec{r}_k) = 36\ A_{2\pi}\ \left(\sqrt{4\pi}\right)^3\sum_{K_2,K_3}\langle 1010|K_2 0\rangle\langle 1010|K_3 0\rangle K_{2\pi}^{K_2,\ell_1,\ell_2}(r_i,r_j)K_{2\pi}^{K_3,\ell_3,\ell_4}(r_i,r_k)\langle\ell_1 0\ell_4 0|L0\rangle$$
$$\times(-)^{(\ell_2+\lambda_2+\ell_3+\lambda_1+\ell_1)}\hat{K}_2\hat{K}_3\hat{\lambda}_2\hat{\lambda}_3\hat{\ell}_1\hat{\ell}_4\left\{\begin{matrix}\lambda_2 & \ell_1 & 1\\ \ell_4 & \lambda_3 & \lambda_1\end{matrix}\right\}\left\{\begin{matrix}\ell_1 & \ell_2 & K_2\\ 1 & 1 & \lambda_2\end{matrix}\right\}\left\{\begin{matrix}\ell_4 & \ell_3 & K_3\\ 1 & 1 & \lambda_3\end{matrix}\right\}$$
$$\times\left[Y^{(\lambda_1)}(\hat{r}_i)\otimes\left[\left[Y^{(\ell_2)}(\hat{r}_j)\otimes\sigma_j^{(1)}\right]^{(\lambda_2)}\otimes\left[Y^{(\ell_3)}(\hat{r}_k)\otimes\sigma_k^{(1)}\right]^{(\lambda_3)}\right]^{(\lambda_1)}\right]^{(0)}\ \langle\vec{\tau}_j\vec{\tau}_k\rangle \tag{1.14}$$

Similar to (3.12) we define a kernel

$$\tilde{K}^{\lambda,\ell_1\ell_2}(r_i,r_j) := \sum_k\ \hat{k}\hat{\lambda}\langle 1010|k0\rangle\left\{\begin{matrix}\ell_1 & \ell_2 & k\\ 1 & 1 & \lambda\end{matrix}\right\}K^{k,\ell_1,\ell_2}(r_i,r_j) \tag{1.15}$$

We leave out the integration part and put it into the wave function by making the replacement:

$$R(x_i)\rightarrow\ x_i\sqrt{w_i}R(x_i) \tag{1.16}$$

We now write the interaction as

$$V^{2\pi,P,C}(\vec{r}_i,\vec{r}_j,\vec{r}_k) = 36\ A_{2\pi}\ \left(\sqrt{4\pi}\right)^3\tilde{K}_{2\pi}^{\lambda_2,\ell_1,\ell_2}(r_i,r_j)\tilde{K}_{2\pi}^{\lambda_3,\ell_3,\ell_4}(r_i,r_k)\langle\ell_1 0\ell_4 0|L0\rangle$$
$$\times(-)^{(\ell_2+\lambda_2+\ell_3+\lambda_1+\ell_1)}\ell_1\hat{\ell}_4\left\{\begin{matrix}\lambda_2 & \ell_1 & 1\\ \ell_4 & \lambda_3 & \lambda_1\end{matrix}\right\}$$
$$\times\left[Y^{(\lambda_1)}(\hat{r}_i)\otimes\left[\left[Y^{(\ell_2)}(\hat{r}_j)\otimes\sigma_j^{(1)}\right]^{(\lambda_2)}\otimes\left[Y^{(\ell_3)}(\hat{r}_k)\otimes\sigma_k^{(1)}\right]^{(\lambda_3)}\right]^{(\lambda_1)}\right]^{(0)}\ \langle\vec{\tau}_j\vec{\tau}_k\rangle \tag{1.17}$$

Now we need to use (0.10) to evaluate the matrix elements.

P-WAVE COMMUTATOR TERM

From our paper on vtni (2.29) with $J=\lambda_1$ and $S=1$ we take it as

$$V^{2\pi,P,C}(\vec{r}_i,\vec{r}_j,\vec{r}_k) = 9\big(\sqrt{4\pi}\big)^3 \sum_{K_2,K_3} \langle 1010|K_2 0\rangle\langle 1010|K_3 0\rangle K_{K2}^{K_2,\ell_1,\ell_2}(r_i,r_j) K_{K3}^{K_3,\ell_3,\ell_4}(r_i,r_k)$$
$$\times(-)^{(\ell_2+\lambda_2+\ell_3+\lambda_3)}\hat{K}_2\hat{K}_3\hat{\lambda}_2\hat{\lambda}_3\hat{\ell}_1\hat{\ell}_4\hat{\lambda}_1\sqrt{3}\begin{Bmatrix}\ell_1 & 1 & \lambda_2\\ \ell_4 & 1 & \lambda_3\\ L & 1 & \lambda_1\end{Bmatrix}\begin{Bmatrix}\ell_1 & \ell_2 & K_2\\ 1 & 1 & \lambda_2\end{Bmatrix}\begin{Bmatrix}\ell_4 & \ell_3 & K_3\\ 1 & 1 & \lambda_3\end{Bmatrix}$$
$$\times\Big[\Big[Y^{(L)}(\hat{r}_i)\otimes\big[\sigma_i^{(1)}\otimes\sigma_i^{(1)}\big]^{(1)}\Big]^{(\lambda_1)}\otimes\Big[\big[Y^{(\ell_2)}(\hat{r}_j)\otimes\sigma_j^{(1)}\big]^{(\lambda_2)}\otimes\big[Y^{(\ell_3)}(\hat{r}_k)\otimes\sigma_k^{(1)}\big]^{(\lambda_3)}\Big]^{(\lambda_1)}\Big]^{(0)} \quad (1.18)$$

Exchanging $\vec{r}_j$ and $\vec{r}_k$ leads to a phase of $(-)^S$. Thus, for the commutator term we have $S=1$ while for the anti-commutator we have $S=0$. Both of those obtain a factor 2 because of the commutator, however, the factor $\frac{1}{4}$ in the strength brings this to a factor $\frac{1}{2}$. Further we use

$$\big[\sigma_i^{(1)}\otimes\sigma_i^{(1)}\big]^{(1)} = \sqrt{2}\,\sigma_i^{(1)} \quad (1.19)$$

Also, we use $\tilde{K}$ as defined in the previous section.

$$V^{2\pi,P,C}(\vec{r}_i,\vec{r}_j,\vec{r}_k) = \sqrt{\frac{3}{2}}\;9\big(\sqrt{4\pi}\big)^3\tilde{K}^{\lambda_2,\ell_1,\ell_2}(r_i,r_j)\tilde{K}^{\lambda_3,\ell_3,\ell_4}(r_i,r_k)$$
$$\times(-)^{(\ell_2+\lambda_2+\ell_3+\lambda_3)}\hat{\ell}_1\hat{\ell}_4\hat{\lambda}_1\begin{Bmatrix}\ell_1 & 1 & \lambda_2\\ \ell_4 & 1 & \lambda_3\\ L & 1 & \lambda_1\end{Bmatrix}$$
$$\times\Big[\big[Y^{(L)}(\hat{r}_i)\otimes\sigma_i^{(1)}\big]^{(\lambda_1)}\otimes\Big[\big[Y^{(\ell_2)}(\hat{r}_j)\otimes\sigma_j^{(1)}\big]^{(\lambda_2)}\otimes\big[Y^{(\ell_3)}(\hat{r}_k)\otimes\sigma_k^{(1)}\big]^{(\lambda_3)}\Big]^{(\lambda_1)}\Big]^{(0)} \quad (1.20)$$

Finally Using this we obtain a sum of three terms

$$\langle p_1\bar{h}_1|V^{X,\lambda}|h_2\bar{p}_2\rangle = 9\sqrt{\frac{3}{2}}(-)^{(k_{p_1}+k_{h_1}+k_h+\ell_2+\ell_3)}\frac{\hat{\lambda}_1\hat{\lambda}_2}{\hat{\lambda}}\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda\\ j_{p_1} & j_{h_1} & j_h\end{Bmatrix}\hat{\ell}_1\hat{\ell}_4\langle\ell_1 0\ell_4 0|L0\rangle$$
$$\times\Big\{(-)^{\lambda_1}\hat{\lambda}_1\begin{Bmatrix}\ell_1 & 1 & \lambda_2\\ \ell_4 & 1 & \lambda\\ L & 1 & \lambda_1\end{Bmatrix}$$
$$\times\frac{\sqrt{4\pi}(-)^{k_h}}{\hat{\lambda}_1}\langle h||\big[Y^{(L)}\sigma\big]^{\lambda_1}||h_1\rangle\frac{\sqrt{4\pi}(-)^{k_{p_1}}}{\hat{\lambda}_2}\langle p_1||\big[Y^{(\ell_2)}\sigma\big]^{\lambda_2}||h\rangle\frac{\sqrt{4\pi}(-)^{k_{p_2}}}{\hat{\lambda}}\langle p_2||\big[Y^{(\ell_3)}\sigma\big]^{\lambda}||h_2\rangle$$
$$\times\tilde{K}^{\lambda_2,\ell_1,\ell_2}(r_i,r_j)\tilde{K}^{\lambda,\ell_3,\ell_4}(r_i,r_k)\;R_h(r_i)R_{h_1}(r_i)R_{p_1}(r_j)R_h(r_j)R_{p_2}(r_k)R_{h_2}(r_k)$$
$$+(-)^{\lambda_2}\hat{\lambda}_2\begin{Bmatrix}\ell_1 & 1 & \lambda\\ \ell_4 & 1 & \lambda_1\\ L & 1 & \lambda_2\end{Bmatrix}$$
$$\times\frac{\sqrt{4\pi}(-)^{k_{p_1}}}{\hat{\lambda}_2}\langle p_1||\big[Y^{(L)}\sigma\big]^{\lambda_2}||h\rangle\frac{\sqrt{4\pi}(-)^{k_{p_2}}}{\hat{\lambda}}\langle p_2||\big[Y^{(\ell_2)}\sigma\big]^{\lambda}||h_2\rangle\frac{\sqrt{4\pi}(-)^{k_h}}{\hat{\lambda}_1}\langle h||\big[Y^{(\ell_3)}\sigma\big]^{\lambda_1}||h_1\rangle$$
$$\times\tilde{K}^{\lambda,\ell_1,\ell_2}(r_i,r_j)\tilde{K}^{\lambda_1,\ell_3,\ell_4}(r_i,r_k)\;R_h(r_i)R_{p_1}(r_i)R_{h_1}(r_k)R_h(r_k)R_{p_2}(r_j)R_{h_2}(r_j)$$
$$(-)^{\lambda}\hat{\lambda}\begin{Bmatrix}\ell_1 & 1 & \lambda_1\\ \ell_4 & 1 & \lambda_2\\ L & 1 & \lambda\end{Bmatrix}$$
$$\times\frac{\sqrt{4\pi}(-)^{k_{p_2}}}{\hat{\lambda}}\langle p_2||\big[Y^{(L)}\sigma\big]^{\lambda}||h_2\rangle\frac{\sqrt{4\pi}(-)^{k_h}}{\hat{\lambda}_1}\langle h||\big[Y^{(\ell_2)}\sigma\big]^{\lambda_1}||h_1\rangle\frac{\sqrt{4\pi}(-)^{k_{p_1}}}{\hat{\lambda}_2}\langle p_1||\big[Y^{(\ell_3)}\sigma\big]^{\lambda_2}||h\rangle$$
$$\times\tilde{K}^{\lambda_1,\ell_1,\ell_2}(r_i,r_j)\tilde{K}^{\lambda_2,\ell_3,\ell_4}(r_i,r_k)\;R_h(r_j)R_{h_1}(r_j)R_{p_1}(r_k)R_h(r_k)R_{p_2}(r_i)R_{h_2}(r_i)\Big\} \quad (1.21)$$

S-WAVE TERM

From Pieper et al. we take it as

$$\sum_{cyc} Z(r_{ij})Z(r_{kj})\vec{\sigma}_i\hat{r}_{ij}\ \vec{\sigma}_k\hat{r}_{kj}$$

with Z composed from the Y and T from the P-wave term as

$$Z(x) = \frac{x}{3}\big[Y(x) - T(x)\big]$$

We need here

$$\vec{\sigma}_i \cdot \hat{r}_{ij} Z(r_{ij}) = 4\pi \sum_{\ell_1,\ell_2} K^{1,\ell_1,\ell_2}_{Zr}(r_i, r_j)\Big[Y^{(\ell_2)}(\hat{r}_j) \otimes \big[Y^{(\ell_1)}(\hat{r}_i) \otimes \sigma^{(1)}_i\big]^{(\ell_2)}\Big]^{(0)} \tag{1.22}$$

We now put the various parts together and recouple

$$V^{2\pi,S}(\vec{r}_i, \vec{r}_j, \vec{r}_k) = \left(\sqrt{4\pi}\,\right)^3 \sum_{\ell_1,\ell_2}\sum_{\ell_3,\ell_4}\sum_{k} K^{1,\ell_1,\ell_2}_{Zr}(r_i, r_j)K^{1,\ell_3,\ell_4}_{Zr}(r_k, r_j)\langle \ell_2 0 \ell_4 0|k0\rangle$$
$$\times\Big[Y^{(k)}(\hat{r}_j) \otimes \Big[\big[Y^{(\ell_1)}(\hat{r}_i) \otimes \sigma^{(1)}_i\big]^{(\ell_2)} \otimes \big[Y^{(\ell_3)}(\hat{r}_k) \otimes \sigma^{(1)}_k\big]^{(\ell_4)}\Big]^{(k)}\Big]^{(0)}$$

where

$$\tilde{Z}r(q) = \int (r\ Z(r))j_1(qr)r^2 dr$$

The **type (a)** matrix element requires $k = 0$ and thus $\ell_2 = \ell_4$.

$$V^{2\pi,S}(\vec{r}_i, \vec{r}_k) = 4\pi \sum_{\ell_1,\ell_2}\sum_{\ell_4} \frac{(-)^{\ell_2}}{\hat{\ell}_2}\int r_j^2 dr_j\ K^{1,\ell_1,\ell_2}_{Zr}(r_i, r_j)K^{1,\ell_3,\ell_2}_{Zr}(r_k, r_j)\sum_h R^2_h(r_j)$$
$$\times\Big[\big[Y^{(\ell_1)}(\hat{r}_i) \otimes \sigma^{(1)}_i\big]^{(\ell_2)} \otimes \big[Y^{(\ell_3)}(\hat{r}_k) \otimes \sigma^{(1)}_k\big]^{(\ell_2)}\Big]^{(0)}$$

We write this in terms of the G's, and use

$$\langle \ell_1 0 \lambda 0|10\rangle\langle \ell_3 0 \lambda 0|10\rangle = 3(-)^{\ell_1+\ell_3+\lambda}\sum_k \begin{Bmatrix} \ell_1 & 1 & \lambda \\ 1 & \ell_3 & k \end{Bmatrix}\langle \ell_1 0 \ell_3 0|k0\rangle\langle 1010|k0\rangle$$

the matrix element is given as

$$\begin{aligned}\langle p_1\bar{h}_1|V^{S,\lambda}|h_2\bar{p}_2\rangle = &\sum_{\ell_1,\ell_3} 3(-)^{\ell_1+\ell_3+\lambda}\sum_k \begin{Bmatrix} \ell_1 & 1 & \lambda \\ 1 & \ell_3 & k \end{Bmatrix}\langle \ell_1 0 \ell_3 0|k0\rangle\langle 1010|k0\rangle \\ &\frac{1}{2\lambda+1}\int r_j^2 dr_j\ G^{1,\ell_1,\lambda}_{Zr}(r_i, r_j)G^{1,\ell_3,\lambda}_{Zr}(r_k, r_j)\sum_h R^2_h(r_j) \\ &\times\Big[\frac{(-)^{k_{p_1}}\sqrt{4\pi}}{\hat{\lambda}}\langle p_1||\big[Y^{(\ell_1)}\sigma_i\big]^{(\lambda)}||h_1\rangle\Big]\Big[\frac{(-)^{k_{p_2}}\sqrt{4\pi}}{\hat{\lambda}}\langle p_2||\big[Y^{(\ell_3)}\sigma_k\big]^{(\lambda)}||h_2\rangle\Big]\end{aligned} \tag{1.23}$$

The iso-spin dependence is identical to that of the anti-commutator term in the $A^{2\pi,P}$. Thus with $k = 0$ this adds to the $\sigma\sigma$ interaction, and with $k = 2$ this adds to the tensor interaction.

The **type (b)** matrix element is obtained as

$$\langle p_1\bar{h}_1|V^{SX,\lambda}|h_2\bar{p}_2\rangle = \left(\sqrt{4\pi}\,\right)^3(-)^{(k_{p_2}+k_{h_1}+\lambda_1+\lambda_2+\lambda)}\frac{\hat{\lambda}_1\hat{\lambda}_2}{\hat{\lambda}}\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda \\ j_{p_1} & j_{h_1} & j_h \end{Bmatrix}$$
$$\times\Big\{\sum_{\ell_1,\ell_2}\sum_{\ell_3,\ell_4}\sum_k (-)^k\ K^{1,\ell_1,\ell_2}_{Zr}(r_i, r_j)K^{1,\ell_3,\ell_4}_{Zr}(r_j, r_k)\langle \ell_2 0 \ell_3 0|k0\rangle$$
$$\frac{1}{\hat{\lambda}_1}\langle h||T^{\lambda_1}_1||h_1\rangle\frac{1}{\hat{\lambda}_2}\langle p_1||T^{\lambda_2}_2||h\rangle\frac{1}{\hat{\lambda}}\langle p_2||T^{\lambda}_3||h_2\rangle$$

$$+\sum_{\ell_1,\ell_2}\sum_{\ell_3,\ell_4}\sum_{k}\ (-)^k\ K^{1,\ell_1,\ell_2}_{Zr}(r_i,r_j)K^{1,\ell_3,\ell_4}_{Zr}(r_j,r_k)\langle\ell_2 0\ell_3 0|k0\rangle$$

$$\frac{1}{\hat{\lambda}_2}\langle p_1||T_1^{\lambda_2}||h\rangle\frac{1}{\hat{\lambda}}\langle p_2||T_2^{\lambda}||h_2\rangle\frac{1}{\hat{\lambda}_1}\langle h||T_3^{\lambda_1}||h_1\rangle$$

$$+\sum_{\ell_1,\ell_2}\sum_{\ell_3,\ell_4}\sum_{k}\ (-)^k\ K^{1,\ell_1,\ell_2}_{Zr}(r_i,r_j)K^{1,\ell_3,\ell_4}_{Zr}(r_j,r_k)\langle\ell_2 0\ell_3 0|k0\rangle$$

$$\frac{1}{\hat{\lambda}}\langle p_2||T_1^{\lambda}||h_2\rangle\frac{1}{\hat{\lambda}_1}\langle h||T_2^{\lambda_1}||h_1\rangle\frac{1}{\hat{\lambda}_2}\langle p_1||T_3^{\lambda_2}||h\rangle\Big\} \tag{1.24}$$

THREE-PI-DELTA TERM

For this term we take the approximate form given in (3.31) as

$$\frac{50}{3}S^I_\sigma S^I_\tau\ +\ \frac{26}{3}A^I_\sigma A^I_\tau \tag{1.25}$$

From the appendix we take this (without the isospin part) as

$$S^I_\sigma = 2y(r_{ij})y(r_{jk})y(r_{ki}) \tag{1.26}$$

$$+\frac{2}{3}\sum_{cyc}\left(r_{ij}^2 t_{ij}y(r_{jk})y(r_{ki})+C_j^2 t(r_{ij})t(r_{jk})y(r_{ki})\right) \tag{1.27}$$

$$-\ \frac{2}{3}C_iC_jC_k t(r_{ij})t(r_{jk})t(r_{ki}) \tag{1.28}$$

$$+\ \Big[\sum_{cyc}\ \vec{\sigma}_i\vec{\sigma}_j\Big]\Big[\frac{2}{3}y_{ij}y_{jk}y_{ki}+\frac{1}{3}\sum_{cyc}r_{ij}^2 t_{ij}y_{jk}y_{ki}\Big] \tag{1.29}$$

$$+\ \frac{1}{3}\sum_{cyc}\vec{\sigma}_i\vec{\sigma}_k C_j^2 t_{ij}t_{jk}y_{ki} \tag{1.30}$$

$$-\ \frac{1}{3}\sum_{cyc}\big(\vec{\sigma}_i\cdot\vec{r}_{ij}\ \vec{\sigma}_j\cdot\vec{r}_{ij}t_{ij}y_{ki}y_{jk}+\vec{\sigma}_i\cdot\vec{r}_{ki}\ \vec{\sigma}_j\cdot\vec{r}_{ki}t_{ki}y_{ij}y_{jk}+\vec{\sigma}_i\cdot\vec{r}_{jk}\ \vec{\sigma}_j\cdot\vec{r}_{jk}t_{jk}y_{ij}y_{ki}\big) \tag{1.31}$$

$$+\ \frac{1}{3}\sum_{cyc}\ C_k\vec{\sigma}_i\cdot\vec{r}_{jk}\vec{\sigma}_j\cdot\vec{r}_{ki}t_{ki}t_{jk}y_{ij} \tag{1.32}$$

$$+\ \frac{1}{3}\sum_{cyc}\ \vec{\sigma}_i\cdot\vec{a}\vec{\sigma}_j\cdot\vec{a}\big(t_{ij}t_{jk}y_{ki}+t_{ij}y_{jk}t_{ki}+C_k t_{ij}t_{jk}t_{ki}\big) \tag{1.33}$$

and

$$A^I_\sigma = \frac{i}{3}\big[\vec{\sigma}_i\cdot\vec{\sigma}_j\times\vec{\sigma}_k y_{ij}y_{jk}y_{ki}+\vec{\sigma}_i\cdot\vec{a}\vec{\sigma}_j\cdot\vec{a}\vec{\sigma}_k\cdot\vec{a}t_{ij}t_{jk}t_{ki}\big] \tag{1.34}$$

$$+\ \frac{i}{3}\sum_{cyc}\ \big(\vec{\sigma}_i\times\vec{\sigma}_j\cdot\vec{r}_{ij}\ \vec{\sigma}_k\cdot\vec{r}_{ij}t_{ij}y_{jk}y_{ki}+\vec{\sigma}_i\cdot\vec{a}\ \vec{\sigma}_j\cdot\vec{\sigma}_k C_i t_{ij}y_{jk}t_{ki}\big) \tag{1.35}$$

$$+\ \frac{i}{3}\sum_{cyc}\ \vec{\sigma}_i\cdot\vec{r}_{jk}\ \vec{\sigma}_k\cdot\vec{r}_{ij}\ \vec{\sigma}_j\cdot\vec{a}t_{ij}t_{jk}y_{ki} \tag{1.36}$$

$$+\ \frac{2i}{3}\sum_{cyc}\ \vec{\sigma}_i\cdot\vec{a}\big(C_i t_{ij}y_{jk}t_{ki}-C_j t_{ij}t_{jk}y_{ki}-C_k y_{ij}t_{jk}t_{ki}-C_jC_k t_{ij}t_{jk}t_{ki}\big) \tag{1.37}$$

where $\vec{a}=\vec{r}_{ij}\times\vec{r}_{jk}=\vec{r}_{jk}\times\vec{r}_{ki}=\vec{r}_{ki}\times\vec{r}_{ij}$ and $C_i=\vec{r}_{ij}\vec{r}_{ik}$.
For this section only three form factors are needed.

$$y_{ij}=4\pi\ \frac{2}{\pi}\int q^2dq\ \tilde{y}(q)\sum_{\ell}(-)^\ell\hat{\ell}\ j_\ell(qr_i)j_\ell(qr_j)\big[Y^{(\ell)}(\hat{r}_i)\otimes Y^{(\ell)}(\hat{r}_j)\big]^{(0)} \tag{1.38}$$

with

$$\tilde{y}(q) = \int r_{ij}^2 dr_{ij} y_{ij} j_0(qr_{ij}) \tag{1.39}$$

then we can write the separated form of this as

$$y_{ij} = 4\pi \sum_{\ell} (-)^{\ell} \hat{\ell} \left[Y^{(\ell)}(\hat{r}_i) \otimes Y^{(\ell)}(\hat{r}_j) \right]^{(0)} K_y^{\ell}(r_i, r_j) \tag{1.40}$$

with

$$K_y^{0,\ell}(r_i, r_j) = \sum_{n,m} H_n^{\ell}(r_i) T_{n,m}^{y} H_m^{\ell}(r_j) \tag{1.41}$$

$$T_{n,m}^{y} = \sum_{i} H_n^{\ell}(q_i) H_m^{\ell}(q_i) \tilde{y}(q_i) q_i^2 w_i \tag{1.42}$$

Here we assume that the Kernel $K_y^{\ell}(r_i, r_j)$ is given on a grid of gauss-points.
We also need the $k = 0, 2$ form factors for $r_{ij}^2 t_{ij}$, and we write:

$$r_{ij}^2 t(r_{ij}) Y^{(k)}(\hat{r}_{ij}) = \sqrt{4\pi} \sum_{\ell_1, \ell_2} K_{r^2 t}^{k, \ell_1, \ell_2}(r_i, r_j) \left[Y^{(\ell_1)}(\hat{r}_i) \otimes Y^{(\ell_2)}(\hat{r}_j) \right]^{(k)}$$

We also note that

$$r_{ij}^2 t_{ij} = 3 T_{ij}$$

and

$$y_{ij} = Y_{ij} - T_{ij}$$

For the **type (a)** matrix elements we have for spectator protons the remaining matrix elements:

$$\begin{aligned} \langle pp | S_{\tau}^{I} | pp \rangle &= 4 \\ \langle pn | S_{\tau}^{I} | pn \rangle &= \frac{4}{3} \\ \langle nn | S_{\tau}^{I} | nn \rangle &= \frac{4}{3} \\ \langle pn | S_{\tau}^{I} | np \rangle &= \frac{4}{3} \end{aligned} \tag{1.43}$$

We now work on each of the 23 terms separately. We first only list the **type (a)** matrix elements for all 23 terms, then we list the **type (b)** matrix elements for all terms.
term S1
The triple product is

$$\begin{aligned} y_{ij} y_{jk} y_{ki} = &\sum_{\ell_1, \ell_2, \ell_3} \sum_{k_1, k_2, k_3} \begin{Bmatrix} k_1 & k_2 & k_3 \\ \ell_2 & \ell_3 & \ell_1 \end{Bmatrix} K_y^{0,\ell_1}(r_i, r_j) \; K_y^{0,\ell_2}(r_j, r_k) \; K_y^{0,\ell_3}(r_k, r_i) \\ &\times (-)^{(\ell_1 + \ell_2 + \ell_3)} (\sqrt{4\pi})^3 \hat{\ell}_1 \hat{\ell}_2 \langle \ell_1 0 \ell_2 0 | k_2 0 \rangle \hat{\ell}_1 \hat{\ell}_3 \langle \ell_1 0 \ell_3 0 | k_1 0 \rangle \hat{\ell}_2 \hat{\ell}_3 \langle \ell_2 0 \ell_3 0 | k_3 0 \rangle \\ &\times \left[Y^{(k_1)}(\hat{r}_i) \otimes \left[Y^{(k_2)}(\hat{r}_j) \otimes Y^{(k_3)}(\hat{r}_k) \right]^{(k_1)} \right]^{(0)} \end{aligned}$$

From this we obtain the **type (a)** matrix element wit $k_1 = 0$ as

$$
\begin{aligned}
\langle p_1\bar{h}_1|V^{eff,\lambda}|h_2\bar{p}_2\rangle = \; & \frac{(-)^\lambda}{\hat{\lambda}} \int r_i^2 dr_i R_h(r_i)R_p(r_i)R_{p_1}(r_j)R_{h_1}(r_j)R_{p_2}(r_k)R_{h_2}(r_k) \\
& \sum_{\ell_2,\ell_3} K_y^{0,\ell_3}(r_i,r_j)\; K_y^{0,\ell_2}(r_j,r_k)\; K_y^{0,\ell_3}(r_k,r_i) \\
& \times \frac{(2\ell_2+1)(2\ell_3+1)}{\hat{\lambda}} \langle \ell_3 0 \ell_2 0|\lambda 0\rangle \langle \ell_3 0 \ell_2 0|\lambda 0\rangle \\
& \times \Big[\frac{(-)^{k_{p_1}}\sqrt{4\pi}}{\hat{\lambda}}\langle p_1||Y_j^\lambda||h_1\rangle\Big]\Big[\frac{(-)^{k_{p_2}}\sqrt{4\pi}}{\hat{\lambda}}\langle p_2||Y_k^\lambda||h_2\rangle\Big] \qquad (1.44)
\end{aligned}
$$

<u>**term S2**</u>

$$
\begin{aligned}
\sum_{cyc} r_{ij}^2 t_{ij} y_{jk} y_{ki} = & \sum_{cyc}\sum_{\ell_1,\ell_2,\ell_3}\sum_{k_1,k_2,k_3} \begin{Bmatrix} k_1 & k_2 & k_3 \\ \ell_2 & \ell_3 & \ell_1 \end{Bmatrix} K_{r^2t}^{0,\ell_1}(r_i,r_j)\; K_y^{0,\ell_2}(r_j,r_k)\; K_y^{0,\ell_3}(r_k,r_i) \\
& \times (-)^{(\ell_1+\ell_2+\ell_3)}(\sqrt{4\pi})^3 \hat{\ell}_1\hat{\ell}_2\langle \ell_1 0 \ell_2 0|k_2 0\rangle \hat{\ell}_1\hat{\ell}_3\langle \ell_1 0 \ell_3 0|k_1 0\rangle \hat{\ell}_2\hat{\ell}_3\langle \ell_2 0 \ell_3 0|k_3 0\rangle \\
& \times \Bigg[Y^{(k_1)}(\hat{r}_i)\otimes\Big[Y^{(k_2)}(\hat{r}_j)\otimes Y^{(k_3)}(\hat{r}_k)\Big]^{(k_1)}\Bigg]^{(0)}
\end{aligned}
$$

we obtain the three **type (a)** matrix elements with $k_1=0$, $k_2=k_3=\lambda$ as

$$
\begin{aligned}
\langle p_1\bar{h}_1|V^{eff,\lambda}|h_2\bar{p}_2\rangle = \; & \frac{(-)^\lambda}{\hat{\lambda}} \int r_i^2 dr_i R_h(r_i)R_p(r_i) \\
& \Big\{ \sum_{\ell_2,\ell_3} K_{r^2t}^{0,\ell_3,\ell_3}(r_i,r_j)\; K_y^{0,\ell_2}(r_j,r_k)\; K_y^{0,\ell_3}(r_k,r_i) \\
& \quad + K_y^{0,\ell_3}(r_i,r_j)\; K_{r^2t}^{0,\ell_2,\ell_2}(r_j,r_k)\; K_y^{0,\ell_3}(r_k,r_i) \\
& \quad + K_y^{0,\ell_3}(r_i,r_j)\; K_y^{0,\ell_2}(r_j,r_k)\; K_{r^2t}^{0,\ell_3,\ell_3}(r_k,r_i)\Big\} \\
& \times \frac{(2\ell_2+1)(2\ell_3+1)}{\hat{\lambda}} \langle \ell_3 0 \ell_2 0|\lambda 0\rangle \langle \ell_3 0 \ell_2 0|\lambda 0\rangle \\
& \times \Big[\frac{(-)^{k_{p_1}}\sqrt{4\pi}}{\hat{\lambda}}\langle p_1||Y_j^\lambda||h_1\rangle\Big]\Big[\frac{(-)^{k_{p_2}}\sqrt{4\pi}}{\hat{\lambda}}\langle p_2||Y_k^\lambda||h_2\rangle\Big] \qquad (1.45)
\end{aligned}
$$

<u>**term S3**</u>

$$
C_j = \vec{r}_{jk}\vec{r}_{ji} = -\vec{r}_{jk}\vec{r}_{ij} = r_{jk}r_{ij}\frac{4\pi}{\sqrt{3}}\Big[Y^{(1)}(\hat{r}_{jk})\otimes Y^{(1)}(\hat{r}_{ij})\Big]^{(0)} \qquad (1.46)
$$

$$
\begin{aligned}
C_j^2 &= \sum_k (\frac{4\pi}{\hat{k}}) r_{jk}^2 r_{ij}^2 (\langle 1010|k0\rangle)^2 \Big[Y^{(k)}(\hat{r}_{jk})\otimes Y^{(k)}(\hat{r}_{ij})\Big]^{(0)} \\
&= \frac{1}{3} r_{jk}^2 r_{ij}^2 + \frac{4\pi}{5} r_{jk}^2 r_{ij}^2 \frac{2}{3}\Big[Y^{(2)}(\hat{r}_{jk})\otimes Y^{(2)}(\hat{r}_{ij})\Big]^{(0)} \qquad (1.47)
\end{aligned}
$$

$$
\begin{aligned}
C_j^2 t_{ij} t_{jk} y_{ki} = & \frac{1}{3} r_{ij}^2 t_{ij} r_{jk}^2 t_{jk} y_{ki} \\
& + \frac{4\pi}{5} r_{ij}^2 t_{ij} r_{jk}^2 t_{jk} y_{ki} \frac{2}{3}\Big[Y^{(2)}(\hat{r}_{jk})\otimes Y^{(2)}(\hat{r}_{ij})\Big]^{(0)} \qquad (1.48)
\end{aligned}
$$

We break this term into the $k=0$ and the $k=2$ contribution. The $k=0$ contribution can be derived immediately from the previous as:

$$
\begin{aligned}
\frac{1}{3} r_{ij}^2 t_{ij} r_{jk}^2 t_{jk} y_{ki} = & \frac{1}{3}\sum_{\ell_1,\ell_2,\ell_3}\sum_{k_1,k_2,k_3} \begin{Bmatrix} k_1 & k_2 & k_3 \\ \ell_2 & \ell_3 & \ell_1 \end{Bmatrix} K_{r^2t}^{0,\ell_1}(r_i,r_j)\; K_{r^2t}^{0,\ell_2}(r_j,r_k)\; K_y^{0,\ell_3}(r_k,r_i) \\
& \times (-)^{(\ell_1+\ell_2+\ell_3)}(\sqrt{4\pi})^3 \hat{\ell}_1\hat{\ell}_2\langle \ell_1 0 \ell_2 0|k_2 0\rangle \hat{\ell}_1\hat{\ell}_3\langle \ell_1 0 \ell_3 0|k_1 0\rangle \hat{\ell}_2\hat{\ell}_3\langle \ell_2 0 \ell_3 0|k_3 0\rangle \\
& \times \Bigg[Y^{(k_1)}(\hat{r}_i)\otimes\Big[Y^{(k_2)}(\hat{r}_j)\otimes Y^{(k_3)}(\hat{r}_k)\Big]^{(k_1)}\Bigg]^{(0)}
\end{aligned}
$$

The $k=2$ contribution is given by

$$\frac{2}{3}\frac{1}{\sqrt{5}}\big(\sqrt{4\pi}\big)^3(-)^{\ell_3+\ell_5+k_3}\hat{\ell}_1\hat{\ell}_2\hat{\ell}_3\hat{\ell}_4\hat{\ell}_5$$
$$\times\langle\ell_2 0\ell_3 0|k_1 0\rangle\langle\ell_1 0\ell_5 0|k_2 0\rangle\langle\ell_4 0\ell_5 0|k_3 0\rangle\begin{Bmatrix}\ell_2 & \ell_1 & 2\\ \ell_4 & \ell_3 & k_1\end{Bmatrix}\begin{Bmatrix}k_2 & k_3 & k_1\\ \ell_4 & \ell_1 & \ell_5\end{Bmatrix}$$
$$\times K^{2,\ell_1,\ell_2}_{r^2t}(r_i,r_j)K^{2,\ell_3,\ell_4}_{r^2t}(r_j,r_k)K^{0,\ell_5}_{y}(r_k,r_i)\Big[Y^{(k_1)}(\hat{r}_j)\otimes\big[Y^{(k_2)}(\hat{r}_i)\otimes Y^{(k_3)}(\hat{r}_k)\big]^{(k_1)}\Big]^{(0)} \tag{1.49}$$

For the $k=0$ contribution we obtain the three **type (a)** matrix elements

$$\langle p_1\bar{h}_1|V^{eff,\lambda}|h_2\bar{p}_2\rangle = \frac{1}{3}\,\frac{(-)^{\lambda}}{\hat{\lambda}}\int r_i^2 dr_i R_h(r_i)R_p(r_i)$$
$$\Big\{\sum_{\ell_2,\ell_3}K^{0,\ell_3,\ell_3}_{r^2t}(r_i,r_j)\;K^{0,\ell_2}_{y}(r_j,r_k)\;K^{0,\ell_3}_{y}(r_k,r_i)$$
$$+K^{0,\ell_3}_{y}(r_i,r_j)\;K^{0,\ell_2,\ell_2}_{r^2t}(r_j,r_k)\;K^{0,\ell_3}_{y}(r_k,r_i)$$
$$+K^{0,\ell_3}_{y}(r_i,r_j)\;K^{0,\ell_2}_{y}(r_j,r_k)\;K^{0,\ell_3,\ell_3}_{r^2t}(r_k,r_i)\Big\}$$
$$\times\;\frac{(2\ell_2+1)(2\ell_3+1)}{\hat{\lambda}}\langle\ell_3 0\ell_2 0|\lambda 0\rangle\langle\ell_3 0\ell_2 0|\lambda 0\rangle$$
$$\times\Big[\frac{(-)^{k_{p_1}}\sqrt{4\pi}}{\hat{\lambda}}\langle p_1||Y^{\lambda}_j||h_1\rangle\Big]\Big[\frac{(-)^{k_{p_2}}\sqrt{4\pi}}{\hat{\lambda}}\langle p_2||Y^{\lambda}_k||h_2\rangle\Big] \tag{1.50}$$

For the $k=2$ contribution we obtain the **type (a)** matrix element as

$$\langle p_1\bar{h}_1|V^{eff,\lambda}|h_2\bar{p}_2\rangle = \frac{2}{3\sqrt{5}}\,\frac{(-)^{\lambda+\ell_2}}{\hat{\lambda}}(\langle\ell_1 0\ell_5 0|\lambda 0\rangle)^2\int r_j^2 dr_j R_h(r_j)R_p(r_j)$$
$$\sum_{\ell_1,\ell_2,\ell_5}K^{2,\ell_1,\ell_2}_{r^2t}(r_i,r_j)\;K^{2,\ell_2,\ell_1}_{r^2t}(r_j,r_k)\;K^{0,\ell_5}_{y}(r_k,r_i)$$
$$\times\Big[\frac{(-)^{k_{p_1}}\sqrt{4\pi}}{\hat{\lambda}}\langle p_1||Y^{\lambda}||h_1\rangle\Big]\Big[\frac{(-)^{k_{p_2}}\sqrt{4\pi}}{\hat{\lambda}}\langle p_2||Y^{\lambda}||h_2\rangle\Big] \tag{1.51}$$

term S4

$$C_jC_k=\sum_k\;\big(\sqrt{4\pi}\big)^3\frac{1}{3}r_{jk}^2r_{ij}r_{ki}\langle 1010|k0\rangle\;\Big[Y^{(k)}(\hat{r}_{jk})\otimes\big[Y^{(1)}(\hat{r}_{ij})\otimes Y^{(1)}(\hat{r}_{ki})\big]^{(k)}\Big]^{(0)} \tag{1.52}$$

$$C_iC_jC_k=\Big(\sqrt{4\pi}\Big)^3r_{jk}^2r_{ki}^2r_{ij}^2\sum_{k_1,k_2,k_3}\begin{Bmatrix}k_1 & k_2 & k_3\\ 1 & 1 & 1\end{Bmatrix}\langle 1010|k_3 0\rangle\langle 1010|k_1 0\rangle\langle 1010|k_2 0\rangle$$
$$\times\;\Big[Y^{(k_3)}(\hat{r}_{jk})\otimes\big[Y^{(k_1)}(\hat{r}_{ij})\otimes Y^{(k_2)}(\hat{r}_{ki})\big]^{(k_3)}\Big]^{(0)} \tag{1.53}$$

$$C_iC_jC_kt_{ij}t_{jk}t_{ki}=\Big(\sqrt{4\pi}\Big)^3\sum_{k_1,k_2,k_3}\begin{Bmatrix}k_1 & k_2 & k_3\\ 1 & 1 & 1\end{Bmatrix}\langle 1010|k_3 0\rangle\langle 1010|k_1 0\rangle\langle 1010|k_2 0\rangle$$
$$\times\;K^{k_3,\ell_1,\ell_2}_{r^2t}(r_j,r_k)\;K^{k_1,\ell_3,\ell_4}_{r^2t}(r_i,r_j)\;K^{k_2,\ell_5,\ell_6}_{r^2t}(r_k,r_i)$$
$$\times\;\Big[\big[Y^{(\ell_1)}(\hat{r}_j)\otimes Y^{(\ell_2)}(\hat{r}_k)\big]^{(k_3)}\otimes\Big[\big[Y^{(\ell_3)}(\hat{r}_i)\otimes Y^{(\ell_4)}(\hat{r}_j)\big]^{(k_1)}\otimes\big[Y^{(\ell_5)}(\hat{r}_k)\otimes Y^{(\ell_6)}(\hat{r}_i)\big]^{(k_2)}\Big]^{(k_3)}\Big]^{(0)}$$
$$=\Big(\sqrt{4\pi}\Big)^3\sum_{k_1,k_2,k_3}\begin{Bmatrix}k_1 & k_2 & k_3\\ 1 & 1 & 1\end{Bmatrix}\langle 1010|k_3 0\rangle\langle 1010|k_1 0\rangle\langle 1010|k_2 0\rangle$$
$$\times\;K^{k_3,\ell_1,\ell_2}_{r^2t}(r_j,r_k)\;K^{k_1,\ell_3,\ell_4}_{r^2t}(r_i,r_j)\;K^{k_2,\ell_5,\ell_6}_{r^2t}(r_k,r_i)$$
$$\hat{k}_1\hat{k}_2\hat{k}_5\hat{k}_3\hat{k}_5\hat{\ell}_1\hat{\ell}_2\hat{\ell}_3\hat{\ell}_4\hat{\ell}_5\hat{\ell}_6\langle\ell_1 0\ell_4 0|k_6 0\rangle\langle\ell_3 0\ell_6 0|k_4 0\rangle\langle\ell_5 0\ell_2 0|k_7 0\rangle$$
$$\times\;\begin{Bmatrix}\ell_3 & \ell_4 & k_1\\ \ell_6 & \ell_5 & k_2\\ k_4 & k_5 & k_3\end{Bmatrix}\begin{Bmatrix}\ell_4 & \ell_5 & k_5\\ \ell_1 & \ell_2 & k_3\\ k_6 & k_7 & k_4\end{Bmatrix}\Big[Y^{(k_4)}(\hat{r}_i)\otimes\big[Y^{(k_6)}(\hat{r}_j)\otimes Y^{(k_7)}(\hat{r}_k)\big]^{(k_4)}\Big]^{(0)} \tag{1.54}$$

As this term is already symmetric under cyclic permutations we have only one term for **type (a)**. This requires $k_4 = 0$ and $k_6 = k_7 = \lambda$.

$$
\begin{aligned}
\langle p_1\bar{h}_1|V^{eff,\lambda}|h_2\bar{p}_2\rangle = \; & \frac{(-)^\lambda}{\hat{\lambda}} \int r_i^2 dr_i R_h(r_i)R_p(r_i)R_{p_1}(r_j)R_{h_1}(r_j)R_{p_2}(r_k)R_{h_2}(r_k) \\
& \sum_{k_1,k_2,k_3} (-)^{k_1+\ell_1+\lambda}\langle 1010|k_30\rangle\langle 1010|k_10\rangle\langle 1010|k_20\rangle\langle \ell_10\ell_40|\lambda 0\rangle\langle \ell_50\ell_20|\lambda 0\rangle \\
& \times \; K_{r^2t}^{k_3,\ell_1,\ell_2}(r_j,r_k)\; K_{r^2t}^{k_1,\ell_3,\ell_4}(r_i,r_j)\; K_{r^2t}^{k_2,\ell_5,\ell_3}(r_k,r_i) \\
& \times \; \begin{Bmatrix} k_1 & k_2 & k_3 \\ 1 & 1 & 1 \end{Bmatrix}\begin{Bmatrix} \ell_4 & k_1 & \ell_3 \\ k_2 & \ell_5 & k_3 \end{Bmatrix}\begin{Bmatrix} \ell_4 & \ell_5 & k_3 \\ \ell_2 & \ell_1 & \lambda \end{Bmatrix}\hat{k}_1\hat{k}_2\hat{k}_3\hat{\ell}_1\hat{\ell}_2\hat{\ell}_4\hat{\ell}_5\frac{1}{\hat{\lambda}} \\
& \times \Big[\frac{(-)^{k_{p_1}}\sqrt{4\pi}}{\hat{\lambda}}\langle p_1||Y^\lambda||h_1\rangle\Big]\Big[\frac{(-)^{k_{p_2}}\sqrt{4\pi}}{\hat{\lambda}}\langle p_2||Y^\lambda||h_2\rangle\Big] \qquad (1.55)
\end{aligned}
$$

term S5

$$
\begin{aligned}
y_{ij}y_{jk}y_{ki}(\vec{\sigma}_j\vec{\sigma}_k) = & \sum_{\ell_1,\ell_2,\ell_3}\sum_{k_1,k_2,k_3}\begin{Bmatrix} k_1 & k_2 & k_3 \\ \ell_2 & \ell_3 & \ell_1 \end{Bmatrix} K_y^{0,\ell_1}(r_i,r_j)\; K_y^{0,\ell_2}(r_j,r_k)\; K_y^{0,\ell_3}(r_k,r_i) \\
& \times (-)^{(\ell_1+k_5)}(\sqrt{4\pi})^3\hat{k}_4\hat{k}_5\hat{\ell}_1\hat{\ell}_2\langle \ell_10\ell_20|k_20\rangle\hat{\ell}_1\hat{\ell}_3\langle \ell_10\ell_30|k_10\rangle\hat{\ell}_2\hat{\ell}_3\langle \ell_20\ell_30|k_30\rangle\begin{Bmatrix} k_4 & k_5 & k_1 \\ k_3 & k_2 & 1 \end{Bmatrix} \\
& \times \Big[Y^{(k_1)}(\hat{r}_i)\otimes\Big[\big[Y^{(k_2)}\sigma_j\big]^{(k_4)}\otimes\big[Y^{(k_3)}\sigma_k\big]^{(k_5)}\Big]^{(k_1)}\Big]^{(0)}
\end{aligned}
$$

We have only one **type (a)** matrix element with $k_1 = 0$ leading to $\ell_1 = \ell_3$ and $k_2 = k_3$

$$
\begin{aligned}
\langle p_1\bar{h}_1|V^{eff,\lambda}|h_2\bar{p}_2\rangle = \; & \frac{(-)^\lambda}{\hat{\lambda}} \int r_i^2 dr_i R_h(r_i)R_p(r_i)R_{p_1}(r_j)R_{h_1}(r_j)R_{p_2}(r_k)R_{h_2}(r_k) \\
& K_y^{0,\ell_1}(r_i,r_j)\; K_y^{0,\ell_2}(r_j,r_k)\; K_y^{0,\ell_1}(r_k,r_i) \\
& (\langle \ell_10\ell_20|k_20\rangle)^2\frac{\hat{\lambda}\hat{\ell}_1\hat{\ell}_1\hat{\ell}_2}{\hat{k}_2}(-)^{k_2+1} \\
& \times\Big[\frac{(-)^{k_{p_1}}\sqrt{4\pi}}{\hat{\lambda}}\langle p_1||\big[Y^{(k_2)}\sigma_j\big]^\lambda||h_1\rangle\Big]\Big[\frac{(-)^{k_{p_2}}\sqrt{4\pi}}{\hat{\lambda}}\langle p_2||\big[Y^{(k_2)}\sigma_k\big]^\lambda||h_2\rangle\Big] \qquad (1.56)
\end{aligned}
$$

term S6

$$
\begin{aligned}
r_{ij}^2t_{ij}y_{jk}y_{ki}\sum_{cyc}\vec{\sigma}_i\vec{\sigma}_k = & \sum_{\ell_1,\ell_2,\ell_3}\sum_{k_1,k_2,k_3}\begin{Bmatrix} k_1 & k_2 & k_3 \\ \ell_2 & \ell_3 & \ell_1 \end{Bmatrix} K_{r^2t}^{0,\ell_1}(r_i,r_j)\; K_y^{0,\ell_2}(r_j,r_k)\; K_y^{0,\ell_3}(r_k,r_i) \\
& \times (-)^{(\ell_1+\ell_2+\ell_3)}(\sqrt{4\pi})^3\hat{\ell}_1\hat{\ell}_2\langle \ell_10\ell_20|k_20\rangle\hat{\ell}_1\hat{\ell}_3\langle \ell_10\ell_30|k_10\rangle\hat{\ell}_2\hat{\ell}_3\langle \ell_20\ell_30|k_30\rangle \\
& \times \Big[Y^{(k_1)}(\hat{r}_i)\otimes\big[Y^{(k_2)}(\hat{r}_j)\otimes Y^{(k_3)}(\hat{r}_k)\big]^{(k_1)}\Big]^{(0)}(\vec{\sigma}_i\vec{\sigma}_j) \\
& + \sum_{\ell_1,\ell_2,\ell_3}\sum_{k_1,k_2,k_3}\begin{Bmatrix} k_1 & k_2 & k_3 \\ \ell_2 & \ell_3 & \ell_1 \end{Bmatrix} K_{r^2t}^{0,\ell_1}(r_i,r_j)\; K_y^{0,\ell_2}(r_j,r_k)\; K_y^{0,\ell_3}(r_k,r_i) \\
& \times (-)^{(\ell_1+\ell_2+\ell_3)}(\sqrt{4\pi})^3\hat{\ell}_1\hat{\ell}_2\langle \ell_10\ell_20|k_20\rangle\hat{\ell}_1\hat{\ell}_3\langle \ell_10\ell_30|k_10\rangle\hat{\ell}_2\hat{\ell}_3\langle \ell_20\ell_30|k_30\rangle \\
& \times \Big[Y^{(k_1)}(\hat{r}_i)\otimes\big[Y^{(k_2)}(\hat{r}_j)\otimes Y^{(k_3)}(\hat{r}_k)\big]^{(k_1)}\Big]^{(0)}(\vec{\sigma}_j\vec{\sigma}_k) \\
& + \sum_{\ell_1,\ell_2,\ell_3}\sum_{k_1,k_2,k_3}\begin{Bmatrix} k_1 & k_2 & k_3 \\ \ell_2 & \ell_3 & \ell_1 \end{Bmatrix} K_{r^2t}^{0,\ell_1}(r_i,r_j)\; K_y^{0,\ell_2}(r_j,r_k)\; K_y^{0,\ell_3}(r_k,r_i) \\
& \times (-)^{(\ell_1+\ell_2+\ell_3)}(\sqrt{4\pi})^3\hat{\ell}_1\hat{\ell}_2\langle \ell_10\ell_20|k_20\rangle\hat{\ell}_1\hat{\ell}_3\langle \ell_10\ell_30|k_10\rangle\hat{\ell}_2\hat{\ell}_3\langle \ell_20\ell_30|k_30\rangle \\
& \times \Big[Y^{(k_1)}(\hat{r}_i)\otimes\big[Y^{(k_2)}(\hat{r}_j)\otimes Y^{(k_3)}(\hat{r}_k)\big]^{(k_1)}\Big]^{(0)}(\vec{\sigma}_k\vec{\sigma}_i)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\ell_1,\ell_2,\ell_3}\sum_{k_1,k_2,k_3} \begin{Bmatrix} k_1 & k_2 & k_3 \\ \ell_2 & \ell_3 & \ell_1 \end{Bmatrix} K^{0,\ell_1}_{r^2t}(r_i,r_j)\ K^{0,\ell_2}_{y}(r_j,r_k)\ K^{0,\ell_3}_{y}(r_k,r_i) \\
&\times (-)^{(\ell_1+k_5+1)}(\sqrt{4\pi})^3\hat{\ell}_1\hat{\ell}_2\langle \ell_1 0 \ell_2 0|k_2 0\rangle \hat{\ell}_1\hat{\ell}_3\langle \ell_1 0 \ell_3 0|k_1 0\rangle \hat{\ell}_2\hat{\ell}_3\langle \ell_2 0 \ell_3 0|k_3 0\rangle \\
&\times \begin{Bmatrix} k_3 & k_2 & k_1 \\ 1 & k_4 & k_5 \end{Bmatrix} \hat{k}_4\hat{k}_5 \left[\left[Y^{(k_1)}\sigma_i\right]^{(k_4)} \otimes \left[\left[Y^{(k_2)}\sigma_j\right]^{(k_5)} \otimes Y^{(k_3)}(\hat{r}_k)\right]^{(k_4)}\right]^{(0)} \\
&\quad + \sum_{\ell_1,\ell_2,\ell_3}\sum_{k_1,k_2,k_3} \begin{Bmatrix} k_1 & k_2 & k_3 \\ \ell_2 & \ell_3 & \ell_1 \end{Bmatrix} K^{0,\ell_1}_{r^2t}(r_i,r_j)\ K^{0,\ell_2}_{y}(r_j,r_k)\ K^{0,\ell_3}_{y}(r_k,r_i) \\
&\times (-)^{(\ell_1+k_5+1)}(\sqrt{4\pi})^3\hat{\ell}_1\hat{\ell}_2\langle \ell_1 0 \ell_2 0|k_2 0\rangle \hat{\ell}_1\hat{\ell}_3\langle \ell_1 0 \ell_3 0|k_1 0\rangle \hat{\ell}_2\hat{\ell}_3\langle \ell_2 0 \ell_3 0|k_3 0\rangle \\
&\times \hat{k}_1\hat{k}_4\hat{k}_5 \begin{Bmatrix} k_4 & k_5 & k_1 \\ k_3 & k_2 & 1 \end{Bmatrix} \left[Y^{(k_1)}(\hat{r}_i) \otimes \left[\left[Y^{(k_2)}\sigma_j\right]^{(k_4)} \otimes \left[Y^{(k_3)}\sigma_k\right]^{(k_5)}\right]^{(k_1)}\right]^{(0)} \\
&\quad + \sum_{\ell_1,\ell_2,\ell_3}\sum_{k_1,k_2,k_3} \begin{Bmatrix} k_1 & k_2 & k_3 \\ \ell_2 & \ell_3 & \ell_1 \end{Bmatrix} K^{0,\ell_1}_{r^2t}(r_i,r_j)\ K^{0,\ell_2}_{y}(r_j,r_k)\ K^{0,\ell_3}_{y}(r_k,r_i) \\
&\times (-)^{(\ell_1+\ell_2+\ell_3+k_2+k_3+k_4)}(\sqrt{4\pi})^3\hat{\ell}_1\hat{\ell}_2\langle \ell_1 0 \ell_2 0|k_2 0\rangle \hat{\ell}_1\hat{\ell}_3\langle \ell_1 0 \ell_3 0|k_1 0\rangle \hat{\ell}_2\hat{\ell}_3\langle \ell_2 0 \ell_3 0|k_3 0\rangle \\
&\times \left[\left[Y^{(k_1)}\sigma_j\right]^{(k_4)} \otimes \left[Y^{(k_2)}(\hat{r}_j) \otimes \left[Y^{(k_3)}\sigma_k\right)\right]^{(k_5)}\right]^{(k_4)}\Big]^{(0)} \qquad (1.57)
\end{aligned}
$$

This results in the three **type (a)** matrix elements:

$$ \qquad (1.58) $$

term S7

We take the spin-independent part from term S3 separated into the $k = 0$ and the $k = 2$ contribution to obtain

$$
\begin{aligned}
\frac{1}{3}r^2_{ij}t_{ij}r^2_{jk}t_{jk}y_{ki}(\vec{\sigma}_i\vec{\sigma}_k) &= \frac{1}{3}\sum_{\ell_1,\ell_2,\ell_3}\sum_{k_1,k_2,k_3} \begin{Bmatrix} k_1 & k_2 & k_3 \\ \ell_2 & \ell_3 & \ell_1 \end{Bmatrix} K^{0,\ell_1}_{r^2t}(r_i,r_j)\ K^{0,\ell_2}_{r^2t}(r_j,r_k)\ K^{0,\ell_3}_{y}(r_k,r_i) \\
&\times (-)^{(\ell_1+\ell_2+\ell_3)}(\sqrt{4\pi})^3\hat{\ell}_1\hat{\ell}_2\langle \ell_1 0 \ell_2 0|k_2 0\rangle \hat{\ell}_1\hat{\ell}_3\langle \ell_1 0 \ell_3 0|k_1 0\rangle \hat{\ell}_2\hat{\ell}_3\langle \ell_2 0 \ell_3 0|k_3 0\rangle \\
&\times \left[Y^{(k_1)}(\hat{r}_i) \otimes \left[Y^{(k_2)}(\hat{r}_j) \otimes Y^{(k_3)}(\hat{r}_k)\right]^{(k_1)}\right]^{(0)} \vec{\sigma}_i\vec{\sigma}_k \\
&= \frac{1}{3}\sum_{\ell_1,\ell_2,\ell_3}\sum_{k_1,k_2,k_3} \begin{Bmatrix} k_1 & k_2 & k_3 \\ \ell_2 & \ell_3 & \ell_1 \end{Bmatrix} K^{0,\ell_1}_{r^2t}(r_i,r_j)\ K^{0,\ell_2}_{r^2t}(r_j,r_k)\ K^{0,\ell_3}_{y}(r_k,r_i) \\
&\times (-)^{(\ell_2+k_4)}(\sqrt{4\pi})^3\hat{\ell}_1\hat{\ell}_2\langle \ell_1 0 \ell_2 0|k_2 0\rangle \hat{\ell}_1\hat{\ell}_3\langle \ell_1 0 \ell_3 0|k_1 0\rangle \hat{\ell}_2\hat{\ell}_3\langle \ell_2 0 \ell_3 0|k_3 0\rangle \\
&\times \begin{Bmatrix} k_2 & k_3 & k_1 \\ 1 & k_4 & k_5 \end{Bmatrix} \hat{k}_4\hat{k}_5 \left[\left[Y^{(k_1)}\sigma_i\right]^{(k_4)} \otimes \left[Y^{(k_2)}(\hat{r}_j) \otimes \left[Y^{(k_3)}\sigma_k\right]^{(k_5)}\right]^{(k_4)}\right]^{(0)} \qquad (1.59)
\end{aligned}
$$

The $k = 2$ contribution is given by

$$
\begin{aligned}
&\frac{2}{3}\frac{1}{\sqrt{5}}\left(\sqrt{4\pi}\right)^3(-)^{k_3+k_5}\hat{\ell}_1\hat{\ell}_2\hat{\ell}_3\hat{\ell}_4\hat{\ell}_5 K^{2,\ell_1,\ell_2}_{r^2t}(r_i,r_j)K^{2,\ell_3,\ell_4}_{r^2t}(r_j,r_k)K^{0,\ell_5}_{y}(r_k,r_i) \\
&\times\langle \ell_2 0 \ell_3 0|k_1 0\rangle\langle \ell_1 0 \ell_5 0|k_2 0\rangle\langle \ell_4 0 \ell_5 0|k_3 0\rangle \begin{Bmatrix} \ell_2 & \ell_1 & 2 \\ \ell_4 & \ell_3 & k_1 \end{Bmatrix}\begin{Bmatrix} k_2 & k_3 & k_1 \\ \ell_4 & \ell_1 & \ell_5 \end{Bmatrix}\begin{Bmatrix} k_4 & k_5 & k_1 \\ k_3 & k_2 & 1 \end{Bmatrix}\hat{k}_4\hat{k}_5 \\
&\times\left[Y^{(k_1)}(\hat{r}_j) \otimes \left[\left[Y^{(k_2)}\sigma_i\right]^{(k_4)} \otimes \left[Y^{(k_3)}\sigma_k\right]^{(k_5)}\right]^{(k_1)}\right]^{(0)} \qquad (1.60)
\end{aligned}
$$

term S8

$$
\vec{\sigma}_i\cdot\vec{r}_{ij}\ \vec{\sigma}_j\cdot\vec{r}_{ki} = 4\pi\ r_{ij}r_{ki}\sum_k \frac{\hat{k}}{3}\left[\left[\sigma^{(1)}_i \otimes \sigma^{(1)}_j\right]^{(k)} \otimes \left[Y^{(1)}(\hat{r}_{ij}) \otimes Y^{(1)}(\hat{r}_{ki})\right]^{(k)}\right]^{(0)} \qquad (1.61)
$$

As a special case of this

$$\vec{\sigma}_i\cdot\vec{r}_{ij}\ \vec{\sigma}_j\cdot\vec{r}_{ij} = \sqrt{4\pi}\ r_{ij}^2\sum_k \langle 1010|k0\rangle\ \left[\left[\sigma_i^{(1)}\otimes\sigma_j^{(1)}\right]^{(k)}\otimes Y^{(k)}(\hat{r}_{ij})\right]^{(0)} \tag{1.62}$$

$$C_k\ \vec{\sigma}_i\cdot\vec{r}_{ij}\ \vec{\sigma}_j\cdot\vec{r}_{ki} = 4\pi r_{ki}^2 r_{jk}^2 \sum_{k_1,k_2,k_3} \hat{k}_3\langle 1010|k_10\rangle\langle 1010|k_20\rangle\begin{Bmatrix} k_1 & k_2 & k_3\\ 1 & 1 & 1\end{Bmatrix}$$
$$\times\left[\left[\sigma_i^{(1)}\otimes\sigma_j^{(1)}\right]^{(k_3)}\otimes\left[Y^{(k_1)}(\hat{r}_{jk})\otimes Y^{(k_2)}(\hat{r}_{ki})\right]^{(k_3)}\right]^{(0)} \tag{1.63}$$

The term has been evaluated as:

$$\vec{\sigma}_i\cdot\vec{r}_{ij}\ \vec{\sigma}_j\cdot\vec{r}_{ij}t_{ij}y_{ki}y_{jk} = \langle 1010|k0\rangle\langle\ell_10\ell_30|k_30\rangle\langle\ell_20\ell_40|k_40\rangle\langle\ell_30\ell_40|k_10\rangle\hat{k}_2\hat{k}_2\hat{k}\hat{k}_5\hat{k}_6\hat{\ell}_1\hat{\ell}_2\hat{\ell}_3\hat{\ell}_4$$
$$\begin{Bmatrix}\ell_1 & \ell_2 & k\\ \ell_3 & \ell_4 & k_1\\ k_3 & k_4 & k_2\end{Bmatrix}\begin{Bmatrix}k_3 & k_4 & k_2\\ 1 & 1 & k\\ k_5 & k_6 & k_1\end{Bmatrix}K_{r^2t}^{k,\ell_1,\ell_2}(r_i,r_j)K_y^{0,\ell_3}(r_i,r_k)K_y^{0,\ell_4}(r_j,r_k)$$
$$\left[Y^{(k_1)}(\hat{r}_k)\otimes\left[\left[Y^{(k_3)}\sigma_i\right]^{(k_5)}\otimes\left[Y^{(k_4)}\sigma_j\right]^{(k_6)}\right]^{(k_1)}\right]^{(0)} \tag{1.64}$$

For the **type (a)** matrix element we set $k_1 = 0$ leading to $k_2 = k$, $k_5 = k_6 = \lambda$, and $\ell_4 = \ell_3$.

$$\vec{\sigma}_i\cdot\vec{r}_{ij}\ \vec{\sigma}_j\cdot\vec{r}_{ij}t_{ij}y_{ki}y_{jk} = \langle 1010|k0\rangle\langle\ell_10\ell_30|k_30\rangle\langle\ell_20\ell_30|k_40\rangle(-)^{\lambda+\ell_1+1}\hat{k}\hat{\lambda}\hat{\ell}_1\hat{\ell}_2$$
$$\begin{Bmatrix}k_3 & k_4 & k\\ 1 & 1 & \lambda\end{Bmatrix}\begin{Bmatrix}k_3 & k_4 & k\\ \ell_2 & \ell_1 & \ell_3\end{Bmatrix}$$
$$K_{r^2t}^{k,\ell_1,\ell_2}(r_i,r_j)K_y^{0,\ell_3}(r_i,r_k)K_y^{0,\ell_3}(r_j,r_k)\left[\left[Y^{(k_3)}\sigma_i\right]^{(\lambda)}\otimes\left[Y^{(k_4)}\sigma_j\right]^{(\lambda)}\right]^{(0)} \tag{1.65}$$

term S9

$$(\vec{\sigma}_i\vec{r}_{ki})\ (\vec{\sigma}_j\vec{r}_{ki})t_{ki}y_{ij}y_{jk} = \sqrt{4\pi}\ \sum_k\langle 1010|k0\rangle\ \left[\left[\sigma_i^{(1)}\otimes\sigma_j^{(1)}\right]^{(k)}\otimes Y^{(k)}(\hat{r}_{ki})\right]^{(0)}(r_{ki}^2t_{ki})y_{ij}y_{jk}$$
$$= \langle 1010|k0\rangle K_{r^2t}^{k,\ell_1,\ell_2}(r_k,r_i)K_y^{0,\ell_3}(r_i,r_j)K_y^{0,\ell_4}(r_j,r_k)$$
$$\hat{k}_2\hat{k}_2\hat{k}\hat{k}_5\hat{k}_6\hat{\ell}_1\hat{\ell}_2\hat{\ell}_3\hat{\ell}_4\langle\ell_20\ell_30|k_10\rangle\langle\ell_10\ell_40|k_30\rangle\langle\ell_20\ell_30|k_40\rangle$$
$$(-)^{k_6+k_4+k_2+k_5}\begin{Bmatrix}k_4 & k_3 & k_2\\ k_5 & 1 & k_6\end{Bmatrix}\begin{Bmatrix}1 & 1 & k\\ k_1 & k_2 & k_5\end{Bmatrix}\begin{Bmatrix}\ell_1 & \ell_2 & k\\ \ell_4 & \ell_3 & k_1\\ k_3 & k_4 & k_2\end{Bmatrix}$$
$$\left[\left[Y^{(k_4)}\sigma_i\right]^{(k_6)}\otimes\left[Y^{(k_3)}(\hat{r}_k)\otimes\left[Y^{(k_1)}\sigma_j\right]^{(k_5)}\right]^{(k_6)}\right]^{(0)} \tag{1.66}$$

For the **type (a)** matrix element we have $k_3 = 0$. This implies that $k_5 = k_6 = \lambda$, $k_4 = k_2$, and $\ell_4 = \ell_1$.

$$(\vec{\sigma}_i\vec{r}_{ki})\ (\vec{\sigma}_j\vec{r}_{ki})t_{ki}y_{ij}y_{jk} = \langle 1010|k0\rangle K_{r^2t}^{k,\ell_1,\ell_2}(r_k,r_i)K_y^{0,\ell_3}(r_i,r_j)K_y^{0,\ell_1}(r_j,r_k)$$
$$\hat{k}\hat{\lambda}\hat{\ell}_2\hat{\ell}_3\langle\ell_20\ell_30|k_10\rangle\langle\ell_20\ell_30|k_20\rangle$$
$$(-)^{k+\ell_3+\lambda+1}\begin{Bmatrix}1 & 1 & k\\ k_1 & k_2 & \lambda\end{Bmatrix}\begin{Bmatrix}k_1 & k & k_2\\ \ell_2 & \ell_3 & \ell_1\end{Bmatrix}$$
$$\left[\left[Y^{(k_2)}\sigma_i\right]^{(\lambda)}\otimes\left[Y^{(k_1)}\sigma_j\right]^{(\lambda)}\right]^{(0)} \tag{1.67}$$

term S10

$$(\vec{\sigma}_i\vec{r}_{jk})\ (\vec{\sigma}_j\vec{r}_{jk})t_{jk}y_{ij}y_{ki} = \sqrt{4\pi}\ \sum_k\langle 1010|k0\rangle\ \left[\left[\sigma_i^{(1)}\otimes\sigma_j^{(1)}\right]^{(k)}\otimes Y^{(k)}(\hat{r}_{jk})\right]^{(0)}(r_{jk}^2t_{jk})y_{ij}y_{ki}$$
$$= \langle 1010|k0\rangle\langle\ell_30\ell_40|k_10\rangle\langle\ell_10\ell_30|k_30\rangle\langle\ell_20\ell_40|k_40\rangle\hat{\ell}_1\hat{\ell}_2\hat{\ell}_3\hat{\ell}_4$$
$$\hat{k}\hat{k}_2\hat{k}_2\hat{k}_5\hat{k}_6(-)^{k_1+k_2}\begin{Bmatrix}k_1 & k_2 & k\\ 1 & 1 & k_5\end{Bmatrix}\begin{Bmatrix}k_4 & k_3 & k_2\\ 1 & k_5 & k_6\end{Bmatrix}\begin{Bmatrix}\ell_3 & \ell_4 & k_1\\ \ell_1 & \ell_2 & k\\ k_3 & k_4 & k_2\end{Bmatrix}$$

$$K^{k,\ell_1,\ell_2}_{r^2t}(r_j,r_k)K^{0,\ell_3}_{y}(r_i,r_j)K^{0,\ell_4}_{y}(r_k,r_i)$$
$$\Big[\big[Y^{(k_1)}\sigma_i\big]^{(k_5)}\otimes\Big[Y^{(k_4)}(\hat r_k)\otimes\big[Y^{(k_3)}\sigma_j\big]^{(k_6)}\Big]^{(k_5)}\Big]^{(0)} \tag{1.68}$$

For the **type (a)** matrix element we have $k_4=0$ which implies $k_6=k_5=\lambda$, $k_3=k_2$, and $\ell_4=\ell_2$.

$$(\vec\sigma_i\vec r_{jk})\ (\vec\sigma_j\vec r_{jk})t_{jk}y_{ij}y_{ki} = \langle 1010|k0\rangle\langle\ell_3 0\ell_2 0|k_1 0\rangle\langle\ell_1 0\ell_3 0|k_2 0\rangle\hat\ell_1\hat\ell_3$$
$$\hat k\hat\lambda(-)^{k_1+\ell_1+\lambda+1}\begin{Bmatrix} k_1 & k_2 & k\\ 1 & 1 & \lambda\end{Bmatrix}\begin{Bmatrix} k_1 & k & k_3\\ \ell_1 & \ell_3 & \ell_2\end{Bmatrix}$$
$$K^{k,\ell_1,\ell_2}_{r^2t}(r_j,r_k)K^{0,\ell_3}_{y}(r_i,r_j)K^{0,\ell_2}_{y}(r_k,r_i)$$
$$\Big[\big[Y^{(k_1)}\sigma_i\big]^{(\lambda)}\otimes\big[Y^{(k_2)}\sigma_j\big]^{(\lambda)}\Big]^{(0)} \tag{1.69}$$

term S11

$$C_k\ (\vec\sigma_i\vec r_{jk})\ (\vec\sigma_j\vec r_{ki})t_{ki}t_{jk}y_{ij} = 4\pi\sum_{k_1,k_2,k_3}(-)^{k_3}\hat k_3\langle 1010|k_1 0\rangle\langle 1010|k_2 0\rangle\begin{Bmatrix} k_1 & k_2 & k_3\\ 1 & 1 & 1\end{Bmatrix}$$
$$\times\Big[\big[\sigma_i^{(1)}\otimes\sigma_j^{(1)}\big]^{(k_3)}\otimes\big[Y^{(k_1)}(\hat r_{jk})\otimes Y^{(k_2)}(\hat r_{ki})\big]^{(k_3)}\Big]^{(0)} r_{ki}^2 t_{ki} r_{jk}^2 t_{jk} y_{ij}$$
$$= (-)^{k_3}\hat k_3\langle 1010|k_1 0\rangle\langle 1010|k_2 0\rangle\begin{Bmatrix} k_1 & k_2 & k_3\\ 1 & 1 & 1\end{Bmatrix}\hat\ell_1\hat\ell_2\hat\ell_3\hat\ell_4\hat\ell_5$$
$$\hat k_1\hat k_2\hat k_3\hat k_4\hat k_4\hat k_8\hat k_9\begin{Bmatrix} k_6 & k_7 & k_4\\ \ell_4 & \ell_1 & \ell_5\end{Bmatrix}\langle\ell_2 0\ell_3 0|k_5 0\rangle\langle\ell_1 0\ell_5 0|k_6 0\rangle\langle\ell_4 0\ell_5 0|k_7 0\rangle$$
$$\begin{Bmatrix} k_6 & k_7 & k_4\\ 1 & 1 & k_3\\ k_8 & k_9 & k_5\end{Bmatrix}\begin{Bmatrix} \ell_1 & \ell_2 & k_1\\ \ell_4 & \ell_3 & k_2\\ k_4 & k_5 & k_3\end{Bmatrix}$$
$$(-)^{\ell_3+k_5+\ell_1}K^{k_1,\ell_1,\ell_2}_{r^2t}(r_j,r_k)K^{k_2,\ell_3,\ell_4}_{r^2t}(r_k,r_i)K^{0,\ell_5}_{y}(r_i,r_j)$$
$$\Big[Y^{(k_5)}(\hat r_k)\otimes\Big[\big[Y^{(k_6)}\sigma_j\big]^{(k_8)}\otimes\big[Y^{(k_7)}\sigma_i\big]^{(k_9)}\Big]^{(k_5)}\Big]^{(0)} \tag{1.70}$$

For the **type (a)** matrix element we have $k_5=0$. This implies $\ell_3=\ell_2$, $k_9=k_8=\lambda$, and $k_4=k_3$. This gives this term as

$$C_k\ (\vec\sigma_i\vec r_{jk})\ (\vec\sigma_j\vec r_{ki})t_{ki}t_{jk}y_{ij} = (-)^{k_3}\hat k_3\langle 1010|k_1 0\rangle\langle 1010|k_2 0\rangle\begin{Bmatrix} k_1 & k_2 & k_3\\ 1 & 1 & 1\end{Bmatrix}\hat\ell_1\hat\ell_4\hat\ell_5$$
$$\hat k_1\hat k_2\hat k_3\hat\lambda\begin{Bmatrix} k_6 & k_7 & k_3\\ \ell_4 & \ell_1 & \ell_5\end{Bmatrix}\langle\ell_1 0\ell_5 0|k_6 0\rangle\langle\ell_4 0\ell_5 0|k_7 0\rangle$$
$$\begin{Bmatrix} k_6 & k_7 & k_3\\ 1 & 1 & \lambda\end{Bmatrix}\begin{Bmatrix} k_1 & k_2 & k_3\\ \ell_4 & \ell_1 & \ell_2\end{Bmatrix}$$
$$(-)^{k_7+k_2+1+\ell_2+\lambda}K^{k_1,\ell_1,\ell_2}_{r^2t}(r_j,r_k)K^{k_2,\ell_2,\ell_4}_{r^2t}(r_k,r_i)K^{0,\ell_5}_{y}(r_i,r_j)$$
$$\Big[\big[Y^{(k_6)}\sigma_j\big]^{(\lambda)}\otimes\big[Y^{(k_7)}\sigma_i\big]^{(\lambda)}\Big]^{(0)} \tag{1.71}$$

term S12

$$(\vec\sigma_i\vec a)\ (\vec\sigma_j\vec a)\ t_{ij}t_{jk}y_{ki} = -6(4\pi)\sum_{k,k_1,k_2}\langle 1010|k_1 0\rangle\langle 1010|k_2 0\rangle\begin{Bmatrix} 1 & 1 & 1\\ 1 & 1 & 1\\ k_1 & k_2 & k\end{Bmatrix}$$
$$\times\Big[\big[\sigma_i^{(1)}\otimes\sigma_j^{(1)}\big]^{(k)}\otimes\big[Y^{(k_1)}(\hat r_{ij})\otimes Y^{(k_2)}(\hat r_{jk})\big]^{(k)}\Big]^{(0)}\ r_{ij}^2 t_{ij} r_{jk}^2 t_{jk} y_{ki}$$
$$= -6\langle 1010|k_1 0\rangle\langle 1010|k_2 0\rangle\begin{Bmatrix} 1 & 1 & 1\\ 1 & 1 & 1\\ k_1 & k_2 & k\end{Bmatrix}\hat k_1\hat k\hat k_2\hat k_3\hat k_3\hat k_7\hat k_8$$
$$\langle\ell_2 0\ell_3 0|k_4 0\rangle\langle\ell_1 0\ell_5 0|k_5 0\rangle\langle\ell_4 0\ell_5 0|k_6 0\rangle\hat\ell_1\hat\ell_2\hat\ell_3\hat\ell_4\hat\ell_5$$

$$\left\{\begin{matrix} k_5 & k_6 & k_3 \\ \ell_4 & \ell_1 & \ell_5 \end{matrix}\right\}\left\{\begin{matrix} k_6 & k_5 & k_3 \\ 1 & k_7 & k_8 \end{matrix}\right\}\left\{\begin{matrix} k_3 & k_4 & k \\ 1 & 1 & k_7 \end{matrix}\right\}\left\{\begin{matrix} \ell_1 & \ell_2 & k_1 \\ \ell_4 & \ell_3 & k_2 \\ k_3 & k_4 & k \end{matrix}\right\}$$
$$(-)^{\ell_3+k_2+\ell_1+k+k_4} K_{r^2t}^{k_1,\ell_1,\ell_2}(r_i,r_j) K_{r^2t}^{k_2,\ell_3,\ell_4}(r_j,r_k) K_y^{0,\ell_5}(r_k,r_i)$$
$$\left[\left[Y^{(k_6)}(\hat{r}_k)\otimes\left[Y^{(k_5)}\sigma_i\right]^{(k_8)}\right]^{(k_7)}\otimes\left[Y^{(k_4)}\sigma_j\right]^{(k_7)}\right]^{(0)} \tag{1.72}$$

For the **type (a)** matrix element we have $k_6 = 0$. This implies $\ell_5 = \ell_4$, $k_7 = k_8 = \lambda$, and $k_5 = k_3$.

$$(\vec{\sigma}_i\vec{a})\ (\vec{\sigma}_j\vec{a})\ t_{ij}t_{jk}y_{ki} = -6\langle 1010|k_10\rangle\langle 1010|k_20\rangle\left\{\begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ k_1 & k_2 & k \end{matrix}\right\}\hat{k}_1\hat{k}\hat{k}_2\hat{\lambda}$$
$$\langle \ell_20\ell_30|k_40\rangle\langle \ell_10\ell_40|k_30\rangle\hat{\ell}_1\hat{\ell}_2\hat{\ell}_3$$
$$\left\{\begin{matrix} k_3 & k_4 & k \\ 1 & 1 & \lambda \end{matrix}\right\}\left\{\begin{matrix} \ell_1 & \ell_2 & k_1 \\ \ell_4 & \ell_3 & k_2 \\ k_3 & k_4 & k \end{matrix}\right\}$$
$$(-)^{\ell_3+k_2+k+k_4+\lambda+1} K_{r^2t}^{k_1,\ell_1,\ell_2}(r_i,r_j) K_{r^2t}^{k_2,\ell_3,\ell_4}(r_j,r_k) K_y^{0,\ell_4}(r_k,r_i)$$
$$\left[\left[Y^{(k_5)}\sigma_i\right]^{(\lambda)}\otimes\left[Y^{(k_4)}\sigma_j\right]^{(\lambda)}\right]^{(0)} \tag{1.73}$$

term S13

$$(\vec{\sigma}_i\vec{a})\ (\vec{\sigma}_j\vec{a})\ t_{ij}y_{jk}t_{ki} = -6(4\pi)\sum_{k,k_1,k_2}\langle 1010|k_10\rangle\langle 1010|k_20\rangle\left\{\begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ k_1 & k_2 & k \end{matrix}\right\}$$
$$\times\left[\left[\sigma_i^{(1)}\otimes\sigma_j^{(1)}\right]^{(k)}\otimes\left[Y^{(k_1)}(\hat{r}_{ki})\otimes Y^{(k_2)}(\hat{r}_{ij})\right]^{(k)}\right]^{(0)}\ r_{ij}^2t_{ij}r_{ki}^2t_{ki}y_{jk}$$
$$= -6\langle 1010|k_10\rangle\langle 1010|k_20\rangle\left\{\begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ k_1 & k_2 & k \end{matrix}\right\}\hat{k}_1\hat{k}_2\hat{k}\hat{k}_4\hat{k}_4\hat{k}_7\hat{k}_8$$
$$\langle \ell_20\ell_30|k_30\rangle\langle \ell_10\ell_50|k_50\rangle\langle \ell_40\ell_50|k_60\rangle\hat{\ell}_1\hat{\ell}_2\hat{\ell}_3\hat{\ell}_4\hat{\ell}_5$$
$$\left\{\begin{matrix} k_3 & k_4 & k \\ 1 & 1 & k_7 \end{matrix}\right\}\left\{\begin{matrix} k_5 & k_6 & k_4 \\ 1 & k_7 & k_8 \end{matrix}\right\}\left\{\begin{matrix} k_5 & k_6 & k_4 \\ \ell_4 & \ell_1 & \ell_5 \end{matrix}\right\}\left\{\begin{matrix} \ell_2 & \ell_1 & k_1 \\ \ell_3 & \ell_4 & k_2 \\ k_3 & k_4 & k \end{matrix}\right\}$$
$$(-)^k K_{r^2t}^{k_1,\ell_1,\ell_2}(r_k,r_i) K_{r^2t}^{k_2,\ell_3,\ell_4}(r_i,r_j) K_y^{0,\ell_5}(r_k,r_j)$$
$$\left[\left[Y^{(k_3)}\sigma_i\right]^{(k_7)}\otimes\left[Y^{(k_5)}(\hat{r}_k)\otimes\left[Y^{(k_6)}\sigma_j\right]^{(k_8)}\right]^{(k_7)}\right]^{(0)} \tag{1.74}$$

For the **type(a)** matrix element we have $k_5 = 0$, $k_7 = k_8 = \lambda$, $\ell_5 = \ell_1$, and $k_6 = k_4$.

$$(\vec{\sigma}_i\vec{a})\ (\vec{\sigma}_j\vec{a})\ t_{ij}y_{jk}t_{ki} = -6\langle 1010|k_10\rangle\langle 1010|k_20\rangle\left\{\begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ k_1 & k_2 & k \end{matrix}\right\}\hat{k}_1\hat{k}_2\hat{k}\hat{\lambda}$$
$$\langle \ell_20\ell_30|k_30\rangle\langle \ell_40\ell_10|k_60\rangle\hat{\ell}_2\hat{\ell}_3\hat{\ell}_4\left\{\begin{matrix} k_3 & k_4 & k \\ 1 & 1 & \lambda \end{matrix}\right\}\left\{\begin{matrix} \ell_2 & \ell_1 & k_1 \\ \ell_3 & \ell_4 & k_2 \\ k_3 & k_4 & k \end{matrix}\right\}$$
$$(-)^{k+\ell_4+\lambda+1} K_{r^2t}^{k_1,\ell_1,\ell_2}(r_k,r_i) K_{r^2t}^{k_2,\ell_3,\ell_4}(r_i,r_j) K_y^{0,\ell_1}(r_k,r_j)$$
$$\left[\left[Y^{(k_3)}\sigma_i\right]^{(\lambda)}\otimes\left[Y^{(k_6)}\sigma_j\right]^{(\lambda)}\right]^{(0)} \tag{1.75}$$

term S14

$$(\vec{\sigma}_i\vec{a})\ (\vec{\sigma}_j\vec{a})C_kt_{ij}t_{jk}t_{ki} = 6\hat{k}\hat{k}_1\langle 1010|k_20\rangle\langle 1010|k_30\rangle\langle 1010|k_40\rangle\left\{\begin{matrix} k_3 & k_4 & k_1 \\ 1 & 1 & 1 \end{matrix}\right\}\left\{\begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ k_1 & k_2 & k \end{matrix}\right\}$$

$$r_{ij}^2 t_{ij} r_{jk}^2 t_{jk} r_{ki}^2 t_{ki}$$

$$\left[\left[\sigma_i^{(1)} \otimes \sigma_j^{(1)}\right]^{(k)} \otimes \left[\left[Y^{(k_3)}(\hat{r}_{ki}) \otimes Y^{(k_4)}(\hat{r}_{jk})\right]^{(k_1)} \otimes Y^{(k_2)}(\hat{r}_{ij})\right]^{(k)}\right]^{(0)}$$

$$= 6\hat{k}\hat{k}_1 \langle 1010|k_2 0\rangle \langle 1010|k_3 0\rangle \langle 1010|k_4 0\rangle \begin{Bmatrix} k_3 & k_4 & k_1 \\ 1 & 1 & 1 \end{Bmatrix} \begin{Bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ k_1 & k_2 & k \end{Bmatrix}$$

$$K_{r^2t}^{k_3,\ell_1,\ell_2}(r_k, r_i) K_{r^2t}^{k_4,\ell_3,\ell_4}(r_j, r_k) K_{r^2t}^{k_2,\ell_5,\ell_6}(r_i, r_j) \hat{\ell}_1 \hat{\ell}_2 \hat{\ell}_3 \hat{\ell}_4 \hat{\ell}_5 \hat{\ell}_6$$

$$(-)^{k+k_2+k_7+k_8} \langle \ell_1 0 \ell_4 0|k_7 0\rangle \langle \ell_2 0 \ell_5 0|k_{10} 0\rangle \langle \ell_3 0 \ell_6 0|k_{11} 0\rangle$$

$$\begin{Bmatrix} k_7 & k_8 & k_1 \\ k_2 & k & k_9 \end{Bmatrix} \begin{Bmatrix} \ell_1 & \ell_2 & k_3 \\ \ell_4 & \ell_3 & k_4 \\ k_7 & k_8 & k_1 \end{Bmatrix} \begin{Bmatrix} \ell_2 & \ell_3 & k_8 \\ \ell_5 & \ell_6 & k_2 \\ k_{10} & k_{11} & k_9 \end{Bmatrix} \begin{Bmatrix} k_{10} & k_{11} & k_9 \\ 1 & 1 & k \\ k_{12} & k_{13} & k_7 \end{Bmatrix}$$

$$\hat{k}_3 \hat{k}_4 \hat{k}_8 \hat{k}_1 \hat{k}_9 \hat{k}_2 \hat{k}_8 \hat{k} \hat{k}_9 \hat{k}_{12} \hat{k}_{13} \left[Y^{(k_7)}(\hat{r}_k) \otimes \left[\left[Y^{(k_{10})}\sigma_i\right]^{(k_{12})} \otimes \left[Y^{(k_{11})}\sigma_j\right]^{(k_{13})}\right]^{(k_7)}\right]^{(0)} \tag{1.76}$$

For the **type (a)** matrix element we set $k_7 = 0$, which implies $k_{12} = k_{13} = \lambda$, $\ell_4 = \ell_1$, $k_9 = k$, and $k_8 = k_1$. For that case we write the interaction

$$(\vec{\sigma}_i \vec{a})\ (\vec{\sigma}_j \vec{a}) C_k t_{ij} t_{jk} t_{ki} = 6\hat{k}\hat{k}_1 \langle 1010|k_2 0\rangle \langle 1010|k_3 0\rangle \langle 1010|k_4 0\rangle \begin{Bmatrix} k_3 & k_4 & k_1 \\ 1 & 1 & 1 \end{Bmatrix} \begin{Bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ k_1 & k_2 & k \end{Bmatrix}$$

$$K_{r^2t}^{k_3,\ell_1,\ell_2}(r_k, r_i) K_{r^2t}^{k_4,\ell_3,\ell_1}(r_j, r_k) K_{r^2t}^{k_2,\ell_5,\ell_6}(r_i, r_j) \hat{\ell}_2 \hat{\ell}_3 \hat{\ell}_5 \hat{\ell}_6$$

$$(-)^{k_3+\ell_3+k_1+k_{11}+1+k+\lambda} \begin{Bmatrix} k_4 & k_3 & k_1 \\ \ell_2 & \ell_3 & \ell_1 \end{Bmatrix} \langle \ell_2 0 \ell_5 0|k_{10} 0\rangle \langle \ell_3 0 \ell_6 0|k_{11} 0\rangle$$

$$\begin{Bmatrix} k_{10} & k_{11} & k \\ 1 & 1 & \lambda \end{Bmatrix} \begin{Bmatrix} \ell_2 & \ell_3 & k_1 \\ \ell_5 & \ell_6 & k_2 \\ k_{10} & k_{11} & k \end{Bmatrix}$$

$$\hat{k}_3 \hat{k}_4 \hat{k}_1 \hat{k}_1 \hat{k}_2 \hat{k} \hat{\lambda} \left[\left[Y^{(k_{10})}\sigma_i\right]^{(\lambda)} \otimes \left[Y^{(k_{11})}\sigma_j\right]^{(\lambda)}\right]^{(0)} \tag{1.77}$$

term A1

We write the first term of A^I as

$$\frac{i}{3}\left[\vec{\sigma}_i \cdot \vec{\sigma}_j \times \vec{\sigma}_k\right] y_{ij} y_{jk} y_{ki} = -(\sqrt{4\pi})^3 \sqrt{\frac{2}{3}} \sum_{\ell_1,\ell_2,\ell_3} \sum_{k_1,k_2,k_3} \begin{Bmatrix} k_1 & k_2 & k_3 \\ \ell_2 & \ell_3 & \ell_1 \end{Bmatrix} K_y^{0,\ell_1}(r_i, r_j)\ K_y^{0,\ell_2}(r_j, r_k)\ K_y^{0,\ell_3}(r_k, r_i)$$

$$\times (-)^{(\ell_1+\ell_2+\ell_3)} \hat{\ell}_1 \hat{\ell}_2 \langle \ell_1 0 \ell_2 0|k_2 0\rangle \hat{\ell}_1 \hat{\ell}_3 \langle \ell_1 0 \ell_3 0|k_1 0\rangle \hat{\ell}_2 \hat{\ell}_3 \langle \ell_2 0 \ell_3 0|k_3 0\rangle$$

$$\times \sum_{k_4,k_5,k_6} (-)^{k_1+k_2+k_3+k_4+k_5+k_6} \hat{k}_4 \hat{k}_5 \hat{k}_6 \begin{Bmatrix} 1 & 1 & 1 \\ k_2 & k_3 & k_1 \\ k_5 & k_6 & k_4 \end{Bmatrix}$$

$$\times \left[\left[Y^{(k_1)}\sigma_i\right]^{(k_4)} \otimes \left[\left[Y^{(k_2)}\sigma_j\right]^{(k_5)} \otimes \left[Y^{(k_3)}\sigma_k\right]^{(k_6)}\right]^{(k_4)}\right]^{(0)}$$

This contributes only to the **type (b)** matrix element. Also, as this is already symmetric under cyclic permutations we need only one term

$$\langle p_1 \bar{h}_1|V^{X,\lambda}|h_2 \bar{p}_2\rangle = (-)^{(k_{p_2}+k_{h_1}+\lambda_1+\lambda_2+\lambda)} \frac{\hat{\lambda}_1 \hat{\lambda}_2}{\hat{\lambda}} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda \\ j_{p_1} & j_{h_1} & j_h \end{Bmatrix}$$

$$\times \left\{ \frac{1}{\hat{\lambda}_1} \langle h||T_1^{\lambda_1}||h_1\rangle \frac{1}{\hat{\lambda}_2} \langle p_1||T_2^{\lambda_2}||h\rangle \frac{1}{\hat{\lambda}} \langle p_2||T_3^{\lambda}||h_2\rangle \right. \tag{1.78}$$

term A2

$$\vec{\sigma}_i \cdot \vec{a} = i\sqrt{\frac{2}{3}}(4\pi) r_{ij} r_{jk} \left[\sigma_i^{(1)} \otimes \left[Y^{(1)}(\hat{r}_{ij}) \otimes Y^{(1)}(\hat{r}_{jk})\right]^{(1)}\right]^{(0)} \tag{1.79}$$

$$\vec{\sigma}_i \cdot \vec{a}\ \vec{\sigma}_j \cdot \vec{a} = -6(4\pi) r_{ij}^2 r_{jk}^2 \sum_{k,k_1,k_2} \langle 1010|k_1 0\rangle \langle 1010|k_2 0\rangle \begin{Bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ k_1 & k_2 & k \end{Bmatrix}$$
$$\times \Big[\big[\sigma_i^{(1)} \otimes \sigma_j^{(1)}\big]^{(k)} \otimes \big[Y^{(k_1)}(\hat{r}_{ij}) \otimes Y^{(k_2)}(\hat{r}_{jk})\big]^{(k)} \Big]^{(0)} \tag{1.80}$$

$$\frac{i}{3}(\vec{\sigma}_i \vec{a})\ (\vec{\sigma}_j \vec{a})\ (\vec{\sigma}_k \vec{a}) t_{ij} t_{jk} t_{ki} = \langle 1010|k_3 0\rangle \langle 1010|k_6 0\rangle \langle 1010|k_7 0\rangle (-)^{k_2+k_3+k_4} \hat{k}_1 \hat{k}_2 \hat{k}_2 \hat{k}_4 \hat{k}_5$$
$$3\sqrt{24} \begin{Bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ k_2 & k_3 & k_1 \end{Bmatrix} \begin{Bmatrix} 1 & 1 & 1 \\ 1 & 1 & k_2 \\ k_6 & k_7 & k_5 \end{Bmatrix} \begin{Bmatrix} k_3 & k_2 & k_1 \\ k_5 & k_4 & 1 \end{Bmatrix}$$
$$\Big[\big[\sigma_i^{(1)} \otimes \sigma_j^{(1)}\big]^{(k_1)} \otimes \Big[\big[\sigma_k^{(1)} \otimes Y^{(k_3)}(\hat{r}_{jk})\big]^{(k_4)} \otimes \big[Y^{(k_6)}(\hat{r}_{ki}) \otimes Y^{(k_7)}(\hat{r}_{ij})\big]^{(k_5)} \Big]^{(k_1)} \Big]^{(0)} \tag{1.81}$$

term A3

$$\frac{i}{3}(\vec{\sigma}_i \times \vec{\sigma}_j \vec{r}_{ij})\ (\vec{\sigma}_k \vec{r}_{ij}) t_{ij} y_{jk} y_{ki} = (-)^{k+1} \frac{\sqrt{2}}{3} \langle 1010|k0\rangle r_{ij}^2 t_{ij} y_{jk} y_{ki}$$
$$\Big[\big[\sigma_i^{(1)} \otimes \sigma_j^{(1)}\big]^{(1)} \otimes \big[\sigma_k^{(1)} \otimes Y^{(k)}(\hat{r}_{ij})\big]^{(1)} \Big]^{(0)}$$
$$= K_{r^2 t}^{k,\ell_1,\ell_2}(r_i, r_j) K_y^{0,\ell_3}(r_j, r_k) K_y^{0,\ell_4}(r_k, r_i)$$
$$(-)^{k+1} \sqrt{2} \langle 1010|k0\rangle \langle \ell_3 0 \ell_4 0|k0\rangle \langle \ell_1 0 \ell_4 0|k_4 0\rangle \langle \ell_2 0 \ell_3 0|k_5 0\rangle \hat{k}_1 \hat{k}_2 \hat{k}_3 \hat{k}_3 \hat{k}_6 \hat{k}_7 \hat{\ell}_1 \hat{\ell}_2 \hat{\ell}_3 \hat{\ell}_4$$
$$(-)^{k_2+k_3+1} \begin{Bmatrix} \ell_1 & \ell_2 & k \\ \ell_4 & \ell_3 & k_1 \\ k_4 & k_5 & k_3 \end{Bmatrix} \begin{Bmatrix} k_4 & k_5 & k_3 \\ 1 & 1 & 1 \\ k_6 & k_7 & k_2 \end{Bmatrix} \begin{Bmatrix} k_2 & k_3 & 1 \\ k & 1 & k_1 \end{Bmatrix}$$
$$\Big[\big[Y^{(k_1)} \sigma_k\big]^{(k_2)} \otimes \Big[\big[Y^{(k_4)} \sigma_i\big]^{(k_6)} \otimes \big[Y^{(k_5)} \sigma_j\big]^{(k_7)} \Big]^{(k_2)} \Big]^{(0)} \tag{1.82}$$

term A4

$$\frac{i}{3}(\vec{\sigma}_i \vec{a})\ (\vec{\sigma}_j \vec{\sigma}_k) C_i t_{ij} y_{jk} t_{ki} = \sqrt{2} \langle 1010|k_1 0\rangle \langle 1010|k_2 0\rangle \hat{k}_2 (-)^{k_2+k_3+k_4+k_6+k_8+1}$$
$$\begin{Bmatrix} k_1 & k_2 & 1 \\ 1 & 1 & 1 \end{Bmatrix} K_{r^2 t}^{k_1,\ell_1,\ell_2}(r_k, r_i) K_{r^2 t}^{k_2,\ell_3,\ell_4}(r_i, r_j) K_y^{0,\ell_5}(r_j, r_k)$$
$$\hat{k}_1 \hat{k}_2 \hat{k}_4 \hat{k}_7 \hat{k}_8 \hat{\ell}_1 \hat{\ell}_2 \hat{\ell}_3 \hat{\ell}_4 \hat{\ell}_5 \langle \ell_1 0 \ell_5 0|k_5 0\rangle \langle \ell_4 0 \ell_5 0|k_6 0\rangle \langle \ell_2 0 \ell_3 0|k_3 0\rangle$$
$$\begin{Bmatrix} k_5 & k_6 & k_4 \\ \ell_4 & \ell_1 & \ell_5 \end{Bmatrix} \begin{Bmatrix} k_7 & k_8 & k_4 \\ k_6 & k_5 & 1 \end{Bmatrix} \begin{Bmatrix} \ell_2 & \ell_1 & k_1 \\ \ell_3 & \ell_4 & k_2 \\ k_3 & k_4 & 1 \end{Bmatrix}$$
$$\Big[\Big[\big[Y^{(k_5)} \sigma_k\big]^{(k_7)} \otimes \big[Y^{(k_6)} \sigma_j\big]^{(k_8)} \Big]^{(k_4)} \otimes \big[Y^{(k_3)} \sigma_i\big]^{(k_4)} \Big]^{(0)} \tag{1.83}$$

term A5

$$\frac{i}{3}(\vec{\sigma}_i \vec{r}_{jk})\ (\vec{\sigma}_k \vec{r}_{ij})\ (\vec{\sigma}_j \vec{a}) t_{ij} t_{jk} y_{ki} = \sqrt{\frac{2}{3}} \hat{k} \hat{k}_1 \langle 1010|k_2 0\rangle \langle 1010|k_3 0\rangle \begin{Bmatrix} 1 & 1 & k \\ 1 & 1 & 1 \\ k_2 & k_3 & k_1 \end{Bmatrix} r_{ij}^2 t_{ij} r_{jk}^2 t_{jk} y_{ki}$$
$$\Big[\Big[\big[\sigma_i^{(1)} \otimes \sigma_k^{(1)}\big]^{(k)} \otimes \sigma_j^{(1)} \Big]^{(k_1)} \otimes \big[Y^{(k_2)}(\hat{r}_{jk}) \otimes Y^{(k_3)}(\hat{r}_{ij})\big]^{(k_1)} \Big]^{(0)} \tag{1.84}$$

term A6

$$\vec{\sigma}_i \cdot \vec{a}\ C_j = i\sqrt{6}\ 4\pi\ r_{ij}^2 r_{jk}^2 \sum_{k_1,k_2} \langle 1010|k_1 0\rangle \langle 1010|k_2 0\rangle (-)^{k_2+1} \hat{k}_2 \begin{Bmatrix} k_1 & k_2 & 1 \\ 1 & 1 & 1 \end{Bmatrix}$$
$$\times \Big[\sigma_i^{(1)} \otimes \big[Y^{(k_1)}(\hat{r}_{ij}) \otimes Y^{(k_2)}(\hat{r}_{jk})\big]^{(1)} \Big]^{(0)} \tag{1.85}$$

The other terms $\vec{\sigma}_i \cdot \vec{a}\ C_k$ and $\vec{\sigma}_i \cdot \vec{a}\ C_i$ can be obtained from this by cyclic permutation (without the σ_i).

$$\vec{\sigma}_i \cdot \vec{a}\ C_j C_k = -i\sqrt{2}\ \left(\sqrt{4\pi}\ \right)^3 r_{ij}^2 r_{jk}^2 r_{ki}^2 \sum_{k,k_1,k_2} \langle 1010|k0\rangle\langle 1010|k_1 0\rangle\langle 1010|k_2\rangle \begin{Bmatrix} k_1 & k_2 & k \\ 1 & 1 & 1 \end{Bmatrix} (-)^{k+k_2}$$
$$\times\left[\sigma_i^{(1)} \otimes \left[Y^{(k)}(\hat{r}_{jk}) \otimes \left[Y^{(k_1)}(\hat{r}_{ij}) \otimes Y^{(k_2)}(\hat{r}_{ki})\right]^{(k)}\right]^{(1)}\right]^{(0)} \tag{1.86}$$

$$\frac{2i}{3}(\vec{\sigma}_i\vec{a})C_i t_{ij} y_{jk} t_{ki} = \frac{2}{3}\sqrt{6}\ 4\pi\ r_{ij}^2 t_{ij} y_{jk} r_{ki}^2 t_{ki} \sum_{k_1,k_2} \langle 1010|k_1 0\rangle\langle 1010|k_2 0\rangle \hat{k}_2 \begin{Bmatrix} k_1 & k_2 & 1 \\ 1 & 1 & 1 \end{Bmatrix}$$
$$\times\left[\sigma_i^{(1)} \otimes \left[Y^{(k_1)}(\hat{r}_{ki}) \otimes Y^{(k_2)}(\hat{r}_{ij})\right]^{(1)}\right]^{(0)}$$
$$= \frac{2}{3}\sqrt{6}\ 4\pi\ K_{r^2t}^{k_1,\ell_1,\ell_2}(r_k,r_i) K_{r^2t}^{k_2,\ell_3,\ell_4}(r_i,r_j) K_y^{0,\ell_5}(r_j,r_k)$$
$$\sum_{k_1,k_2} \langle 1010|k_1 0\rangle\langle 1010|k_2 0\rangle \hat{k}_2 \begin{Bmatrix} k_1 & k_2 & 1 \\ 1 & 1 & 1 \end{Bmatrix}$$
$$\times\langle \ell_1 0 \ell_5 0|k_5 0\rangle\langle \ell_2 0 \ell_3 0|k_3 0\rangle\langle \ell_4 0 \ell_5 0|k_6 0\rangle \hat{\ell}_1\hat{\ell}_2\hat{\ell}_3\hat{\ell}_4\hat{\ell}_5\hat{k}_1\hat{k}_2\hat{k}_3\hat{k}_4$$
$$(-)^{\ell_2+\ell_5+k_1+k_4+k_6} \begin{Bmatrix} k_5 & k_6 & k_4 \\ \ell_4 & \ell_1 & \ell_5 \end{Bmatrix} \begin{Bmatrix} \ell_2 & \ell_1 & k_1 \\ \ell_3 & \ell_4 & k_2 \\ k_3 & k_4 & 1 \end{Bmatrix}$$
$$\times\left[\left[Y^{(k_5)}(\hat{r}_k) \otimes Y^{(k_6)}(\hat{r}_j)\right]^{(k_4)} \otimes \left[Y^{(k_3)}\sigma_i\right]^{(k_4)}\right]^{(0)} \tag{1.87}$$

term A7

$$-\frac{2i}{3}(\vec{\sigma}_i\vec{a})\ C_j t_{ij} t_{jk} y_{ki} = -\frac{2}{3}\sqrt{6}\ 4\pi\ r_{ij}^2 t_{ij} r_{jk}^2 t_{jk} y_{ki} \sum_{k_1,k_2} \langle 1010|k_1 0\rangle\langle 1010|k_2 0\rangle \hat{k}_2 \begin{Bmatrix} k_1 & k_2 & 1 \\ 1 & 1 & 1 \end{Bmatrix}$$
$$\times\left[\sigma_i^{(1)} \otimes \left[Y^{(k_1)}(\hat{r}_{ij}) \otimes Y^{(k_2)}(\hat{r}_{jk})\right]^{(1)}\right]^{(0)}$$
$$= -2\sqrt{\frac{2}{3}}\langle 1010|k_1 0\rangle\langle 1010|k_2 0\rangle \hat{k}_2 \begin{Bmatrix} k_1 & k_2 & 1 \\ 1 & 1 & 1 \end{Bmatrix} \langle \ell_1 0 \ell_2 0|k_3 0\rangle\langle \ell_1 0 \ell_5 0|k_7 0\rangle$$
$$\langle \ell_3 0 \ell_4 0|k_6 0\rangle \begin{Bmatrix} k_2 & 1 & k_1 \\ k_3 & k_4 & k_5 \end{Bmatrix} \begin{Bmatrix} k_3 & k_4 & k_1 \\ \ell_3 & \ell_2 & \ell_1 \end{Bmatrix} \begin{Bmatrix} \ell_3 & \ell_1 & k_4 \\ \ell_4 & \ell_5 & k_2 \\ k_6 & k_7 & k_5 \end{Bmatrix}$$
$$\hat{\ell}_1\hat{\ell}_2\hat{\ell}_3\hat{\ell}_4\hat{\ell}_5\hat{k}_1\hat{k}_2\hat{k}_4\hat{k}_4\hat{k}_5(-)^{k_2+k_3+k_5} K_y^{0,\ell_1}(r_k,r_i) K_{r^2t}^{k_1,\ell_2,\ell_3}(r_i,r_j) K_{r^2t}^{k_2,\ell_4,\ell_5}(r_j,r_k)$$
$$\left[\left[Y^{(k_6)}(\hat{r}_j) \otimes Y^{(k_7)}(\hat{r}_k)\right]^{(k_5)} \otimes \left[Y^{(k_3)}\sigma_i\right]^{(k_5)}\right]^{(0)} \tag{1.88}$$

term A8

$$-\frac{2i}{3}(\vec{\sigma}_i\vec{a})\ C_k y_{ij} t_{jk} t_{ki} = -\frac{2}{3}\sqrt{6}\ 4\pi\ y_{ij} r_{jk}^2 t_{jk} r_{ki}^2 t_{ki} \sum_{k_1,k_2} \langle 1010|k_1 0\rangle\langle 1010|k_2 0\rangle \hat{k}_2 \begin{Bmatrix} k_1 & k_2 & 1 \\ 1 & 1 & 1 \end{Bmatrix}$$
$$\times\left[\sigma_i^{(1)} \otimes \left[Y^{(k_1)}(\hat{r}_{jk}) \otimes Y^{(k_2)}(\hat{r}_{ki})\right]^{(1)}\right]^{(0)}$$
$$= 2\sqrt{\frac{2}{3}}\langle 1010|k_1 0\rangle\langle 1010|k_2 0\rangle \hat{k}_2 \begin{Bmatrix} k_1 & k_2 & 1 \\ 1 & 1 & 1 \end{Bmatrix} \begin{Bmatrix} k_3 & k_4 & k_2 \\ 1\ell_5 & \ell_4 & \ell_1 \end{Bmatrix} \begin{Bmatrix} k_4 & k_3 & k_2 \\ k_1 & 1 & k_5 \end{Bmatrix}$$
$$(-)^{k_5+k_6+k_7+1}\hat{\ell}_5\hat{\ell}_4\hat{\ell}_3\hat{\ell}_2\hat{\ell}_1\hat{k}_1\hat{k}_2\hat{k}_2\hat{k}_5 \langle \ell_5 0 \ell_1 0|k_4 0\rangle\langle \ell_4 0 \ell_3 0|k_6 0\rangle\langle \ell_1 0 \ell_2 0|k_7 0\rangle \begin{Bmatrix} \ell_4 & \ell_1 & k_2 \\ \ell_3 & \ell_2 & k_1 \\ k_6 & k_7 & k_5 \end{Bmatrix}$$
$$K_y^{0,\ell_1}(r_i,r_j) K_{r^2t}^{k_1,\ell_2,\ell_3}(r_j,r_k) K_{r^2t}^{k_2,\ell_4,\ell_5}(r_k,r_i)$$
$$\left[\left[Y^{(k_6)}(\hat{r}_k) \otimes Y^{(k_7)}(\hat{r}_j)\right]^{(k_5)} \otimes \left[Y^{(k_4)}\sigma_i\right]^{(k_5)}\right]^{(0)} \tag{1.89}$$

term A9

$$-\frac{2i}{3}(\vec{\sigma}_i\vec{a})\ C_jC_kt_{ij}t_{jk}t_{ki} = -\frac{2}{3}\sqrt{2}\ \left(\sqrt{4\pi}\ \right)^3 \sum_{k,k_1,k_2} \langle 1010|k0\rangle\langle 1010|k_10\rangle\langle 1010|k_2\rangle \left\{ \begin{array}{ccc} k_1 & k_2 & k \\ 1 & 1 & 1 \end{array} \right\}$$

$$\times\Big[\sigma_i^{(1)} \otimes \Big[Y^{(k)}(\hat{r}_{jk}) \otimes \left[Y^{(k_1)}(\hat{r}_{ij}) \otimes Y^{(k_2)}(\hat{r}_{ki})\right]^{(k)}\Big]^{(1)}\Big]^{(0)} r_{ij}^2t_{ij}r_{jk}^2t_{jk}r_{ki}^2t_{ki}$$

$$= \frac{2}{3}\sqrt{2}\ \left(\sqrt{4\pi}\ \right)^3 \sum_{k,k_1,k_2} \langle 1010|k0\rangle\langle 1010|k_10\rangle\langle 1010|k_2\rangle \left\{ \begin{array}{ccc} k_1 & k_2 & k \\ 1 & 1 & 1 \end{array} \right\}$$

$$\times K_{r^2t}^{k,\ell_1,\ell_2}(r_j,r_k)K_{r^2t}^{k_1,\ell_3,\ell_4}(r_i,r_j)K_{r^2t}^{k_2,\ell_5,\ell_6}(r_k,r_i)$$

$$(-)^{k_5+k_4}\hat{\ell}_1\hat{\ell}_2\hat{\ell}_3\hat{\ell}_4\hat{\ell}_5\hat{\ell}_6\langle \ell_30\ell_60|k_30\rangle\langle \ell_10\ell_40|k_60\rangle\langle \ell_20\ell_40|k_70\rangle\hat{k}\hat{k}\hat{k}_1\hat{k}_2\hat{k}_4\hat{k}_4\hat{k}_5$$

$$\times\left\{ \begin{array}{ccc} k_3 & k_4 & k \\ k & 1 & k_5 \end{array} \right\}\left\{ \begin{array}{ccc} \ell_3 & \ell_4 & k_1 \\ \ell_6 & \ell_5 & k_2 \\ k_3 & k_4 & k \end{array} \right\}\left\{ \begin{array}{ccc} \ell_4 & \ell_5 & k_4 \\ \ell_1 & \ell_2 & k \\ k_6 & k_7 & 1 \end{array} \right\}$$

$$\Big[\left[Y^{(k_3)}\sigma_i\right]^{(k_5)} \otimes \left[Y^{(k_6)}(\hat{r}_j) \otimes Y^{(k_7)}(\hat{r}_k)\right]^{(k_5)}\Big]^{(0)} \qquad (1.90)$$

TYPE B MATRIX ELEMENTS: THREE-PI-DELTA TERM

term S1

The **type (b)** matrix element is

$$\langle p_1\bar{h}_1|V^{X,\lambda}|h_2\bar{p}_2\rangle = \sum_{\lambda_1,\lambda_2} (-)^{(k_{p_1}+k_{h_1}+k_h+\lambda_1+\lambda_2+\lambda)}\frac{\hat{\lambda}_1\hat{\lambda}_2}{\hat{\lambda}}\left\{ \begin{array}{ccc} \lambda_1 & \lambda_2 & \lambda \\ j_{p_1} & j_{h_1} & j_h \end{array} \right\}$$

$$\sum_{\ell_1,\ell_2,\ell_3} \left\{ \begin{array}{ccc} \lambda & \lambda_1 & \lambda_2 \\ \ell_2 & \ell_3 & \ell_1 \end{array} \right\}(-)^{(\ell_1+\ell_2+\ell_3)}\hat{\ell}_1\hat{\ell}_2\langle \ell_10\ell_20|\lambda_10\rangle\hat{\ell}_1\hat{\ell}_3\langle \ell_10\ell_30|\lambda 0\rangle\hat{\ell}_2\hat{\ell}_3\langle \ell_20\ell_30|\lambda_20\rangle$$

$$\times\ K_y^{0,\ell_1}(r_i,r_j)\ K_y^{0,\ell_2}(r_j,r_k)\ K_y^{0,\ell_3}(r_k,r_i)R_h(r_i)R_{h_1}(r_i)R_{p_1}(r_j)R_h(r_j)R_{p_2}(r_k)R_{h_2}(r_k)$$

$$\times\Big[\frac{(-)^{k_h}\sqrt{4\pi}}{\hat{\lambda}_1}\langle h||Y_1^{\lambda_1}||h_1\rangle\Big]\Big[\frac{(-)^{k_{p_1}}\sqrt{4\pi}}{\hat{\lambda}_2}\langle p_1||Y_2^{\lambda_2}||h\rangle\Big]\Big[\frac{(-)^{k_{p_2}}\sqrt{4\pi}}{\hat{\lambda}}\langle p_2||Y_3^{\lambda}||h_2\rangle\Big] \qquad (1.91)$$